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Abstract. We study the pruning process developed by Abraham and Delmas (2012) on the discrete Galton-Watson sub-trees of the Lévy tree which are obtained by considering the minimal sub-tree connecting the root and leaves chosen uniformly at rate λ, see Duquesne and Le Gall (2002). The tree-valued process, as λ increases, has been studied by Duquesne and Winkel (2007). Notice that we have a tree-valued process indexed by two parameters the pruning parameter θ and the intensity λ. Our main results are: construction and marginals of the pruning process, representation of the pruning process (forward in time that is as θ increases) and description of the growing process (backward in time that is as θ decreases) and distribution of the ascension time (or explosion time of the backward process) as well as the tree at the ascension time. A by-product of our result is that the super-critical Lévy trees independently introduced by Abraham and Delmas (2012) and Duquesne and Winkel (2007) coincide. This work is also related to the pruning of discrete Galton-Watson trees studied by Abraham, Delmas and He (2012).

1. Introduction

The study of pruning of Galton-Watson trees has been initiated by Aldous and Pitman [10]. Roughly speaking, it corresponds to the percolation on edges: an edge is uniformly chosen at random in the Galton-Watson tree and it is removed and only the connected component containing the root remains. This procedure is then iterated. This process can be extended backward in time. It corresponds then to a non-decreasing tree-valued process. The ascension time A, is then the first time at which this tree-valued process reaches an unbounded tree. In [10], the authors give the joint distribution of A as well as the tree just before the ascension time (in backward time). The limits of Galton-Watson trees are the so called continuum Lévy trees, see [7, 8, 17, 13]; they are characterized by a branching mechanism ψ which is also a Lévy exponent. The result for the pruning process on Galton-Watson trees was then extended by Abraham and Delmas [2] to a process indexed by time θ whose marginals are continuum Lévy trees. In the setting of the Brownian continuum random tree, which corresponds to a quadratic branching mechanism, the pruning procedure is uniform on the skeleton, see also Aldous and Pitman [9] for a fragmentation point of view in this case. This is the analogue of [10]. However in the general Lévy case, one has to take into account the pruning of nodes with a rate given by its "size" or "mass", which is defined as the asymptotic number of small trees attached to the node. This result in the continuous setting motivated a new pruning procedure on the nodes of Galton-Watson trees, which was developed by Abraham, Delmas and He [3]. In this case, the pruning happens on the nodes with rate depending on the degree of the nodes.

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In the present work, we study the pruning process developed in [2] on the discrete Galton-Watson sub-tree of the Lévy tree. The discrete Galton-Watson sub-trees of the Lévy trees are obtained by considering the minimal sub-tree connecting the root and leaves chosen uniformly with rate $\lambda \geq 0$, see Duquesne and Le Gall [13]. The tree-valued process, as λ increases, has been studied by Duquesne and Winkel [14], in particular to construct super-critical Lévy trees. Notice that super-critical Lévy trees have also been defined in [2]. One of the by-product of our results is that the two definitions coincide, see Section 5. Notice that we have a tree-valued process indexed by two parameters θ (as in [10, 2]) and λ (as in [14]). The other main results are: construction and marginals of the pruning process in Section 4, representation of the pruning process (forward in time that is as θ increases) and description of the growing process (backward in time that is as θ decreases) in Section 6, some remarks on martingales related to the number of leaves in Section 7, distributions of the ascension time and of the tree at the ascension time in Section 8.

Now, we present more precisely our results. Let ψ be a branching mechanism satisfying some regularity conditions (see (H1-3) in Section 2.6). We define ψ_θ by:

$$\psi_\theta(q) = \psi(q+\theta) - \psi(\theta) \quad \text{for all } q \geq 0,$$

and set Θ^ψ the set of θ for which ψ_θ is well defined. We consider the tree-valued process $(\mathcal{T}_\theta, \theta \in \Theta^\psi)$ introduced in [2], corresponding to a uniform pruning on the skeleton and to a pruning at nodes with rate depending on its size. We recall that $\mathcal{T}_\theta$ is a Lévy tree with branching mechanism ψ_θ. Let $\mathbf{m}^{\mathcal{T}_\theta}$ be its mass measure, which is a uniform measure on the set of leaves. Let $\tau_0(\lambda)$ be the minimal sub-tree of $\mathcal{T}_0$ generated by the root and leaves chosen before time λ according to a Poisson point measure $\mathcal{P}^0$ on $\mathbb{R}_+ \times \mathcal{T}_0$ with intensity $dt\, \mathbf{m}^{\mathcal{T}_\theta}$. Let M_λ be the number of chosen leaves: $M_\lambda = \mathcal{P}^0([0,\lambda] \times \mathcal{T}_0)$, so that $\tau_0(\lambda)$ is well defined for $M_\lambda \geq 1$. And we set $\tau_\theta(\lambda) = \mathcal{T}_\theta \bigcap \tau_0(\lambda)$ for $\theta \geq 0$. So we get a two-parameter family of sub-trees $(\mathcal{T}_\theta(\lambda), \lambda \geq 0, \theta \geq 0)$. Let $\mathbb{P}^{\psi,\lambda}$ be the conditional probability given the event $\{M_\lambda \geq 1\}$. We will be interested in the process $\lambda \mapsto \tau_\theta(\lambda)$ which was studied in [14] and in the pruning process $\theta \mapsto \tau_\theta(\lambda)$, which for $\lambda = +\infty$ was studied in [2].

Notice that the leaves of $\tau_\theta(\lambda)$ correspond to marked leaves belonging to $\mathcal{T}_\theta$ as well as roots of sub-trees of $\mathcal{T}_0$ with marked leaves which are removed to get $\mathcal{T}_\theta$. If one is interested only in $\hat{\tau}_\theta(\lambda)$, the minimal sub-tree containing the root and the marked leaves belonging to $\mathcal{T}_\theta$, then one would get a process such that $\hat{\tau}_\theta(\lambda)$ has the same distribution as $\tau_\theta(\lambda_\theta)$ with $\lambda_\theta = \psi(\psi_\theta^{-1}(\lambda)))$. This would lead to another natural process index by the level-set of the function $(\theta, \lambda) \mapsto \psi_\theta(\psi^{-1}(\lambda))$.

Theorem 3.2.1 in [12] in the sub-critical case and Corollary 4.5 in this paper in the general case gives that the sub-tree $\tau_\theta(\lambda)$ is distributed as a Galton-Watson real tree; its reproduction law has generating function $g_{(\psi_\theta, \psi_\theta(\eta))}$, see definition (28), with $\eta = \psi^{-1}(\lambda)$ and exponential individual lifetime with parameter $\psi_\theta'(\eta)$. If we endow $\mathcal{T}_\theta$ with its mass measure and $\tau_\theta(\lambda)$ with a discrete mass measure defined by

$$\mathbf{m}^{\tau_\theta(\lambda)} = \frac{1}{\psi_\theta(\eta)} \sum_{x \text{ a leaf of } \tau_\theta(\lambda)} \delta_x,$$

then we have in Theorem 5.1 the convergence for the Gromov-Hausdorff-Prohorov distance defined in [5] of $\tau_\theta(\lambda)$ to $\mathcal{T}_\theta$ as λ goes to infinity. This result was already in [14] (with the Gromov-Hausdorff distance instead of the Gromov-Hausdorff-Prohorov distance), and this insures that in the super-critical case the Lévy trees introduced in [14] and in [2] are the same. We give in Theorems 6.1 and 6.3 a precise description of the process $(\tau_\theta(\lambda), \theta \geq 0)$ in forward (decreasing tree-valued process) and backward (increasing tree-valued process) time. By considering the backward process, we see it is possible to extend the process up to θ_λ backward in time, with θ_λ defined roughly by $\psi_\theta(\psi^{-1}(\lambda)) = 0$ (see (41) for a precise definition). Usually θ_λ is not the lower bound of Θ^ψ. Intuitively, when θ decreases, the tree grows and in order to keep the right number of leaves, the intensity for choosing them has to decrease; this can be done up to the lower bound θ_λ.

By considering $L_\theta(\lambda)$ the number of leaves of $\tau_\theta(\lambda)$, we get that $\psi'(\theta)L_\theta(\lambda)/\psi_q(\psi^{-1}(\lambda))$ is a backward martingale, see Proposition 7.2. By taking the limit as λ goes to infinity, and since the total mass of $\mathbf{m}^{\tau_\theta(\lambda)}$, that is $L_\theta(\lambda)/\psi_q(\psi^{-1}(\lambda))$, converges to the total mass of $\mathbf{m}^{\mathcal{T}_\theta}$, say σ_θ, we get in Proposition 7.1 that $\psi'(\theta)\sigma_\theta$ is also a backward martingale.

Then we consider the process $(\tau_\theta(\lambda), \theta > \theta_\lambda)$ backward in time and consider its ascension time A_λ defined in (48) as the first time at which the tree $\tau_\theta(\lambda)$ is unbounded. Of course, this corresponds to the ascension time of $(\mathcal{T}_\theta, \theta \in \Theta^\psi)$ when it is larger than θ_λ. We give in Proposition 8.1 the distribution of $(\tau_\theta(\lambda), \theta \geq A_\lambda)$ and identify it in Proposition 8.11 using the pruning of a tree $\mathcal{T}_0^*(\lambda)$ with an infinite spine defined in Sections 8.2 and 8.3. We also prove the convergence, as λ goes to infinity of the tree $\mathcal{T}_0^*(\lambda)$ toward the CRT $\mathcal{T}_0^*$ with infinite spine introduced in [2]. The latter can be seen as a sub-tree of Lévy trees with immigration, see [11] for further work in this direction.

2. Lévy trees and the forest obtained by pruning

2.1. **Notations.** Let (E, d) be a metric Polish space. We denote by $\mathcal{M}_f(E)$ (resp. $\mathcal{M}_f^{\mathrm{loc}}(E)$) the space of all finite (resp. locally finite) Borel measures on E. For $x \in E$, let δ_x denote the Dirac measure at point x. For $\mu \in \mathcal{M}_f^{\mathrm{loc}}(E)$ and f a non-negative measurable function, we set $\langle \mu, f\rangle = \int f(x)\,\mu(dx) = \mu(f)$.

2.2. **Real trees.** We refer to [15] or [16] for a general presentation of random real trees. Informally, real trees are metric spaces without loops, locally isometric to the real line. More precisely, a metric space (T, d) is a real tree if the following properties are satisfied:

(1) For every $s, t \in T$, there is a unique isometric map $f_{s,t}$ from $[0, d(s,t)]$ to T such that $f_{s,t}(0) = s$ and $f_{s,t}(d(s,t)) = t$.
(2) For every $s, t \in T$, if q is a continuous injective map from $[0, 1]$ to T such that $q(0) = s$ and $q(1) = t$, then $q([0, 1]) = f_{s,t}([0, d(s,t)])$.

If $s, t \in T$, we will note $[\![s, t]\!]$ the range of the isometric map $f_{s,t}$ described above and $[\![s, t[\![$ for $[\![s, t]\!]\backslash\{t\}$.

We say that $(T, d, \emptyset)$ is a rooted real tree with root $\emptyset$ if (T, d) is a real tree and $\emptyset \in T$ is a distinguished vertex.

Let $(T, d, \emptyset)$ be a rooted real tree. The degree $n(x)$ of $x \in T$ is the number of connected components of $T \setminus \{x\}$ and the number of children of $x \neq \emptyset$ is $\kappa_x = n(x) - 1$ and of the root is $\kappa_\emptyset = n(\emptyset)$. We shall consider the set of leaves $\mathrm{Lf}(T) = \{x \in T, \kappa_x = 0\}$, the set of branching points $\mathrm{Br}(T) = \{x \in T, \kappa_x \geq 2\}$ and the set of infinite branching points $\mathrm{Br}_\infty(T) = \{x \in T, \kappa_x = \infty\}$. We say that a tree is discrete if $\{x \in \mathrm{Lf}(T) \cup \mathrm{Br}(T); d(\emptyset, x) \leq a\}$ is finite for all a. The skeleton of T is the set of points in the tree that aren't leaves: $\mathrm{Sk}(T) = T\backslash\mathrm{Lf}(T)$. The trace of the Borel σ-field of T restricted to $\mathrm{Sk}(T)$ is generated by the sets $[\![s, s']\!]$; $s, s' \in \mathrm{Sk}(T)$. One defines uniquely a σ-finite Borel measure ℓ^T on T, called the length measure of T, such that:

$$\ell^T(\mathrm{Lf}(T)) = 0 \quad \text{and} \quad \ell^T([\![s, s']\!]) = d(s, s').$$

For every $x \in T$, $[\![\emptyset, x]\!]$ is interpreted as the ancestral line of vertex x in the tree. We define a partial order on T by setting $x \preccurlyeq y$ (x is an ancestor of y) if $x \in [\![\emptyset, y]\!]$. If $x, y \in T$, there exists a unique $z \in T$, called the Most Recent Common Ancestor (MRCA) of x and y, such that $[\![\emptyset, x]\!] \cap [\![\emptyset, y]\!] = [\![\emptyset, z]\!]$, and we write $z = x \wedge y$.

2.3. **Measured rooted real trees.** According to [5], one can define a Gromov-Hausdorff-Prohorov metric on the space of rooted measured metric space as follows.

Let (X, d) be a Polish metric space. For $A, B \in \mathcal{B}(X)$, we set:

$$d_{\mathrm{H}}(A, B) = \inf\{\varepsilon > 0,\ A \subset B^\varepsilon \text{ and } B \subset A^\varepsilon\},$$

the Hausdorff distance between A and B, where $A^\varepsilon = \{x \in X, \inf_{y \in A} d(x, y) < \varepsilon\}$ is the ε-halo set of A. If $\mu, \nu \in \mathcal{M}_f(X)$, we set:

$$d_{\mathrm{P}}(\mu, \nu) = \inf\{\varepsilon > 0,\ \mu(A) \leq \nu(A^\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A^\varepsilon) + \varepsilon \ \text{ for all closed set } A\},$$

the Prohorov distance between μ and ν.

A rooted measured metric space $\mathcal{X} = (X, d, \emptyset, \mu)$ is a metric space (X, d) with a distinguished element $\emptyset \in X$ and a locally finite Borel measure $\mu \in \mathcal{M}_f^{\mathrm{loc}}(E)$. Two rooted measured metric spaces $\mathcal{X} = (X, d, \emptyset, \mu)$ and $\mathcal{X}' = (X', d', \emptyset', \mu')$ are said GHP-isometric if there exists an isometric bijection $\Phi : X \to X'$ such that $\Phi(\emptyset) = \emptyset'$ and $\Phi_*\mu = \mu'$, where $\Phi_*\mu$ is the measure μ transported by Φ.

Let $\mathcal{X} = (X, d, \emptyset, \mu)$ and $\mathcal{X}' = (X', d', \emptyset', \mu')$ be two compact rooted measured metric spaces, and define:

$$d^c_{\mathrm{GHP}}(\mathcal{X}, \mathcal{X}') = \inf_{\Phi, \Phi', Z} \left(d_{\mathrm{H}}^Z(\Phi(X), \Phi'(X')) + d^Z(\Phi(\emptyset), \Phi'(\emptyset')) + d_{\mathrm{P}}^Z(\Phi_*\mu, \Phi'_*\mu') \right),$$

where the infimum is taken over all isometric embeddings $\Phi : X \hookrightarrow Z$ and $\Phi' : X' \hookrightarrow Z$ into some common Polish metric space (Z, d^Z).

If $\mathcal{X} = (X, d, \emptyset, \mu)$ is a rooted measured metric space, then for $r \geq 0$ we will consider its restriction to the ball of radius r centered at $\emptyset$, $\mathcal{X}^{(r)} = (X^{(r)}, d^{(r)}, \emptyset, \mu^{(r)})$, where

$$X^{(r)} = \{x \in X; d(\emptyset, x) \leq r\},$$

the metric $d^{(r)}$ is the restriction of d to $X^{(r)}$, and the measure $\mu^{(r)}(dx) = \mathbf{1}_{X^{(r)}}(x)\ \mu(dx)$ is the restriction of μ to $X^{(r)}$.

We will denote by $\mathbb{T}$ the set of (GHP-isometry classes of) measured rooted real trees $(T, d, \emptyset, \mathbf{m})$ where $(T, d, \emptyset)$ is a locally compact rooted real tree and $\mathbf{m} \in \mathcal{M}_f^{\mathrm{loc}}(T)$ is a locally finite measure on T. Sometimes, we will write $(T, d^T, \emptyset^T, \mathbf{m}^T)$ for $(T, d, \emptyset, \mathbf{m})$ to stress the dependence in T. Sometimes, when there is no confusion, we will simply write T for $(T, d, \emptyset, \mathbf{m})$ and $\tilde{T}$ for $(T, d, \emptyset)$. We define the following function on $\mathbb{T}^2$, for $T_1, T_2 \in \mathbb{T}$:

$$d_{\mathrm{GHP}}(T_1, T_2) = \int_0^\infty \mathrm{e}^{-r} \left(1 \wedge d^c_{\mathrm{GHP}}\left(T_1^{(r)}, T_2^{(r)}\right)\right)\ dr.$$

According to Corollary 2.8 in [4], the function d_{GHP} is well defined and $(\mathbb{T}, d_{\mathrm{GHP}})$ is a Polish metric space.

2.4. **Grafting procedure.** We will define in this section a procedure by which we add (graft) measured rooted real trees on an existing measured rooted real trees. More precisely, let T be a measured rooted real tree and let $((T_i, x_i), i \in I)$ be a finite or countable family of elements of $\mathbb{T} \times T$. We define the real tree obtained by grafting the trees T_i on T at point x_i. We set $\hat{T} = T \sqcup \left(\bigsqcup_{i \in I} T_i \backslash \{\emptyset^{T_i}\}\right)$ where the symbol $\sqcup$ means that we choose for the sets $(T_i)_{i \in I}$ representatives of GHP-isometry classes in $\mathbb{T}$ which are disjoint subsets of some common set and that we perform the disjoint union of all these sets. We set $\emptyset^{\hat{T}} = \emptyset^{T}$. The set $\hat{T}$ is endowed with the following metric $d^{\hat{T}}$: if $s, t \in \hat{T}$,

$$d^{\hat{T}}(s,t) = \begin{cases} d^T(s,t) & \text{if } s,t \in T, \\ d^T(s,x_i) + d^{T_i}(\emptyset^{T_i}, t) & \text{if } s \in T,\ t \in T_i\backslash\{\emptyset^{T_i}\}, \\ d^{T_i}(s,t) & \text{if } s,t \in T_i\backslash\{\emptyset^{T_i}\}, \\ d^T(x_i,x_j) + d^{T_j}(\emptyset^{T_j}, s) + d^{T_i}(\emptyset^{T_i}, t) & \text{if } i \neq j \text{ and } s \in T_j\backslash\{\emptyset^{T_j}\},\ t \in T_i\backslash\{\emptyset^{T_i}\}. \end{cases}$$

We define the mass measure on $\hat{T}$ by:

$$\mathbf{m}^{\hat{T}} = \mathbf{m}^T + \sum_{i \in I} \left(\mathbf{1}_{T_i\backslash\{\emptyset^{T_i}\}} \mathbf{m}^{T_i} + \mathbf{m}^{T_i}(\{\emptyset^{T_i}\})\delta_{x_i} \right).$$

It is clear that the rooted metric space $(\hat{T}, d^{\hat{T}}, \emptyset^{\hat{T}})$ is still a rooted complete real tree. (Notice that it is not always true that $\hat{T}$ remains locally compact or that $\mathbf{m}^{\hat{T}}$ defines a locally finite measure on $\hat{T}$). We will use the following notation for the grafted tree:

$$T \circledast_{i \in I} (T_i, x_i) = (\hat{T}, d^{\hat{T}}, \emptyset^{\hat{T}}, \mathbf{m}^{\hat{T}}), \tag{1}$$

where we make the convention that $T \circledast_{i \in I} (T_i, x_i) = T$ for $I = \emptyset$. If φ is an isometry from T onto T', then $T \circledast_{i \in I} (T_i, x_i)$ and $T' \circledast_{i \in I} (T_i, \varphi(x_i))$ are also isometric. Therefore, the grafting procedure is well defined on $\mathbb{T}$.

In Section 3.2, we shall use the grafting procedure for rooted real trees but without mass measure. Recall $\tilde{T} = (T, \emptyset^T, d^T)$. We shall use the following notation:

$$\tilde{T} \tilde{\circledast}_{i \in I} (\tilde{T}_i, x_i) = (\hat{T}, d^{\hat{T}}, \emptyset^{\hat{T}}), \tag{2}$$

where we also make the convention that $\tilde{T} \tilde{\circledast}_{i \in I} (\tilde{T}_i, x_i) = \tilde{T}$ for $I = \emptyset$.

2.5. **Sub-trees above a given level.** For $T \in \mathbb{T}$ we set $H_{\max}(T) = \sup_{x \in T} d^T(\emptyset^T, x)$ the height of T and for $a \geq 0$:

$$T^{(a)} = \{x \in T,\ d(\emptyset, x) \leq a\} \quad \text{and} \quad T(a) = \{x \in T,\ d(\emptyset, x) = a\}$$

the restriction of the tree T under level a and the set of vertices of T at level a respectively. We denote by $(T^{i,\circ}, i \in I)$ the connected components of $T \setminus T^{(a)}$. Let $\emptyset_i$ be the MRCA of all the vertices of $T^{i,\circ}$. We consider the real tree $T^i = T^{i,\circ} \cup \{\emptyset_i\}$ rooted at point $\emptyset_i$ with mass measure $\mathbf{m}^{T^i}$ defined as the restriction of $\mathbf{m}^T$ to $T^{i,\circ}$. Notice that $T = T^{(a)} \circledast_{i \in I} (T_i, \emptyset_i)$. We will consider the point measure on $T \times \mathbb{T}$:

$$\mathcal{N}_a^T = \sum_{i \in I} \delta_{(\emptyset_i, T^i)}. \tag{3}$$

2.6. **Excursion measure of a Lévy tree.** Let $\alpha \in \mathbb{R}$, $\beta \geq 0$ and π be a σ-finite measure on $(0, +\infty)$ such that $\int_{(0,+\infty)} (r \wedge r^2) \pi(dr) < +\infty$. The branching mechanism ψ with characteristic (α, β, π) is defined by:

$$\psi(\lambda) = \alpha\lambda + \beta\lambda^2 + \int_{(0,+\infty)} \left(\mathrm{e}^{-\lambda r} - 1 + \lambda r \right) \pi(dr). \tag{4}$$

We assume the following assumptions:

(H1) The branching mechanism ψ is conservative: for all $\varepsilon > 0$,

$$\int_0^\varepsilon \frac{d\lambda}{|\psi(\lambda)|} = +\infty.$$

The conservative assumption is equivalent to the finiteness of the corresponding CSBP at all time.

(H2) The Grey condition holds:

$$\int^{+\infty} \frac{d\lambda}{\psi(\lambda)} < +\infty. \tag{5}$$

The Grey condition is equivalent to the a.s. finiteness of the extinction time of the corresponding CSBP. This assumption is used to ensure that the corresponding Lévy tree is compact.

(H3) $\beta > 0$ or $\int_{(0,1)} \ell\pi(d\ell) = +\infty$. This condition is equivalent to the fact that the Lévy process with index ψ is of infinite variation (and the Lévy tree is not discrete).

Let v be the unique non-negative solution of the equation:

$$\int_{v(a)}^{+\infty} \frac{d\lambda}{\psi(\lambda)} = a.$$

Results from [13] in the (sub)critical case, using the coding of compact real trees by height function, can be extended to the super-critical case, see [4]. They can be stated in the following form. There exists a σ-finite measure $\mathbb{N}^\psi[d\mathcal{T}]$ on $\mathbb{T}$, or excursion measure of a Lévy tree, with the following properties.

(i) **Height.** For all $a > 0$, $\mathbb{N}^\psi[H_{\max}(\mathcal{T}) > a] = v(a)$.

(ii) **Mass measure.** The mass measure $\mathbf{m}^\mathcal{T}$ is supported by $\mathrm{Lf}(\mathcal{T})$, $\mathbb{N}^\psi[d\mathcal{T}]$-a.e.

(iii) **Local time.** There exists a $\mathcal{T}$-measure valued process $(\ell^a, a \geq 0)$ càdlàg for the weak topology on finite measure on $\mathcal{T}$ such that $\mathbb{N}^\psi[d\mathcal{T}]$-a.e.:

$$\mathbf{m}^\mathcal{T}(dx) = \int_0^\infty \ell^a(dx)\, da, \tag{6}$$

$\ell^0 = 0$, $\inf\{a > 0; \ell^a = 0\} = \sup\{a \geq 0; \ell^a \neq 0\} = H_{\max}(\mathcal{T})$ and for every fixed $a \geq 0$, $\mathbb{N}^\psi[d\mathcal{T}]$-a.e.:

- The measure ℓ^a is supported on $\mathcal{T}(a)$.
- We have for every bounded continuous function ϕ on $\mathcal{T}$:

$$\begin{aligned}\langle \ell^a, \phi\rangle &= \lim_{\varepsilon\downarrow 0} \frac{1}{v(\varepsilon)} \int \phi(x)\mathbf{1}_{\{H_{\max}(\mathcal{T}')\geq\varepsilon\}} \mathcal{N}_a^\mathcal{T}(dx, d\mathcal{T}')\\ &= \lim_{\varepsilon\downarrow 0} \frac{1}{v(\varepsilon)} \int \phi(x)\mathbf{1}_{\{H_{\max}(\mathcal{T}')\geq\varepsilon\}} \mathcal{N}_{a-\varepsilon}^\mathcal{T}(dx, d\mathcal{T}'), \text{ if } a > 0.\end{aligned}$$

Under $\mathbb{N}^\psi$, the real valued process $(\langle \ell^a, 1\rangle, a \geq 0)$ is distributed as a CSBP with branching mechanism ψ under its canonical measure.

(iv) **Branching property.** For every $a > 0$, the conditional distribution of the point measure $\mathcal{N}_a^\mathcal{T}(dx, d\mathcal{T}')$ under $\mathbb{N}^\psi[d\mathcal{T}|H_{\max}(\mathcal{T}) > a]$, given $\mathcal{T}^{(a)}$, is that of a Poisson point measure on $\mathcal{T}(a) \times \mathbb{T}$ with intensity $\ell^a(dx)\mathbb{N}^\psi[d\mathcal{T}']$.

(v) **Branching points.**

- $\mathbb{N}^\psi[d\mathcal{T}]$-a.e., the branching points of $\mathcal{T}$ have 2 children or an infinity number of children.

- The set of binary branching points (i.e. with 2 children) is empty $\mathbb{N}^\psi$ a.e if $\beta = 0$ and is a countable dense subset of $\mathcal{T}$ if $\beta > 0$.
- The set $\mathrm{Br}_\infty(\mathcal{T})$ of infinite branching points is nonempty with $\mathbb{N}^\psi$-positive measure if and only if $\pi \neq 0$. If $\langle \pi, 1\rangle = +\infty$, the set $\mathrm{Br}_\infty(\mathcal{T})$ is $\mathbb{N}^\psi$-a.e. a countable dense subset of $\mathcal{T}$.

(vi) **Mass of the nodes.** The set $\{d(\emptyset, x),\ x \in \mathrm{Br}_\infty(\mathcal{T})\}$ coincides $\mathbb{N}^\psi$-a.e. with the set of discontinuity times of the mapping $a \mapsto \ell^a$. Moreover, $\mathbb{N}^\psi$-a.e., for every such discontinuity time b, there is a unique $x_b \in \mathrm{Br}_\infty(\mathcal{T}) \cap \mathcal{T}(b)$ and $\Delta_b > 0$, such that:

$$\ell^b = \ell^{b-} + \Delta_b \delta_{x_b},$$

where $\Delta_b > 0$ is called the mass of the node x_b. Furthermore Δ_b can be obtained by the approximation:

$$\Delta_b = \lim_{\varepsilon \to 0} \frac{1}{v(\varepsilon)} n(x_b, \varepsilon), \tag{7}$$

where $n(x_b, \varepsilon) = \int \mathbf{1}_{\{x_b\}}(x) \mathbf{1}_{\{H_{\max}(\mathcal{T}') > \varepsilon\}} \mathcal{N}_b^{\mathcal{T}}(dx, d\mathcal{T}')$ is the number of sub-trees with MRCA x_b and height larger than ε.

In order to stress the dependence in $\mathcal{T}$, we may write $\ell^{a,\mathcal{T}}$ for ℓ^a. We set $\sigma^{\mathcal{T}}$ or simply σ when there is no confusion, for the total mass of the mass measure on $\mathcal{T}$:

$$\sigma = \mathbf{m}^{\mathcal{T}}(\mathcal{T}). \tag{8}$$

Notice that (6) readily implies that $\mathbf{m}^{\mathcal{T}}(\{x\}) = 0$ for all $x \in \mathcal{T}$.

2.7. **Related measure on Lévy trees.** We define a probability measure on $\mathbb{T}$ as follow. Let $r > 0$ and $\sum_{k \in \mathcal{K}} \delta_{\mathcal{T}^k}$ be a Poisson point measure on $\mathbb{T}$ with intensity $r\mathbb{N}^\psi$. Consider $\emptyset$ as the trivial measured rooted real tree reduced to the root with null mass measure. Define $\mathcal{T} = \emptyset \circledast_{k \in \mathcal{K}} (\mathcal{T}^k, \emptyset)$. Using Property (i) as well as (10) below, one easily get that $\mathcal{T}$ is a measured locally compact rooted real tree, and thus belongs to $\mathbb{T}$. We denote by $\mathbb{P}_r^\psi$ its distribution. Its corresponding local time and mass measure are respectively defined by $\ell^a = \sum_{k \in \mathcal{K}} \ell^{a, \mathcal{T}^k}$ for $a \geq 0$, and $\mathbf{m}^{\mathcal{T}} = \sum_{k \in \mathcal{K}} \mathbf{m}^{\mathcal{T}^k}$. Furthermore, its total mass is defined by $\sigma = \sum_{k \in \mathcal{K}} \sigma^{\mathcal{T}^k}$. By construction, we have $\mathbb{P}_r^\psi(d\mathcal{T})$-a.s. $\emptyset \in \mathrm{Br}_\infty(\mathcal{T})$, $\Delta_\emptyset = r$ (see definition (7) with $b = 0$) and $\ell^0 = r\delta_\emptyset$. Under $\mathbb{P}_r^\psi$ or under $\mathbb{N}^\psi$, we define the process $\mathcal{Z} = (\mathcal{Z}_a, a \geq 0)$ by:

$$\mathcal{Z}_a = \langle \ell^a, 1 \rangle.$$

According to Property (iii), under $\mathbb{P}_r^\psi$ (resp. $\mathbb{N}^\psi$), the real valued process $\mathcal{Z}$ is distributed as a CSBP with branching mechanism ψ with initial value r (resp. under its canonical measure). Notice that (under $\mathbb{N}$ or $\mathbb{P}_r^\psi$):

$$\sigma = \int_0^{+\infty} \mathcal{Z}_a \, da = \mathbf{m}^{\mathcal{T}}(\mathcal{T}). \tag{9}$$

In particular, as σ is distributed as the total mass of a CSBP under its canonical measure, we have that $\mathbb{N}^\psi$-a.s. $\sigma > 0$ and for $q > 0$ such that $\psi(q) > 0$:

$$\mathbb{N}^\psi \left[1 - \mathrm{e}^{-\psi(q)\sigma} \right] = q \quad \text{and} \quad \mathbb{N}^\psi \left[\sigma \, \mathrm{e}^{-\psi(q)\sigma} \right] = \frac{1}{\psi'(q)}. \tag{10}$$

The last equation holds for $q = 0$ if $\psi'(0) > 0$.

We will consider the following measures on $\mathbb{T}$:

$$\mathcal{N}_\theta^\psi[d\mathcal{T}] = 2\beta\theta\mathbb{N}^\psi[d\mathcal{T}] + \int_{(0,+\infty)} \pi(dr)(1-\mathrm{e}^{-\theta r})\mathbb{P}_r^\psi(d\mathcal{T}) \tag{11}$$

and

$$\mathbf{N}^\psi[d\mathcal{T}] = \frac{\partial}{\partial\theta}\mathcal{N}_\theta^\psi[d\mathcal{T}]_{|\theta=0} = 2\beta\mathbb{N}^\psi[d\mathcal{T}] + \int_0^{+\infty} r\pi(dr)\,\mathbb{P}_r^\psi(d\mathcal{T}). \tag{12}$$

Elementary computations yield for $q>0$ such that $\psi(q)>0$:

$$\mathcal{N}_\theta^\psi\left[1-\mathrm{e}^{-\psi(q)\sigma}\right] = \psi(\theta+q)-\psi(\theta)-\psi(q) \quad\text{and}\quad \mathbf{N}^\psi\left[1-\mathrm{e}^{-\psi(q)\sigma}\right] = \psi'(q)-\psi'(0). \tag{13}$$

2.8. **Girsanov transformation.** For $\theta\in\mathbb{R}$, we set $\pi_\theta(dr) = \mathrm{e}^{-\theta r}\,\pi(dr)$. Let Θ' be the set of $\theta\in\mathbb{R}$ such that $\int_{(1,+\infty)}\pi_\theta(dr)<+\infty$. If $\pi=0$, then $\Theta'=\mathbb{R}$. We also set $\theta_\infty = \inf\Theta'$. It is obvious that $[0,+\infty)\subset\Theta'$, $\theta_\infty\leq 0$ and either $\Theta'=[\theta_\infty,+\infty)$ or $\Theta'=(\theta_\infty,+\infty)$. We introduce the following branching mechanisms ψ_θ for $\theta\in\Theta'$:

$$\psi_\theta(\lambda) = \psi(\lambda+\theta)-\psi(\theta), \quad \lambda+\theta\in\Theta', \tag{14}$$

with characteristic:

$$\left(\psi'(\theta),\beta,\pi_\theta\right). \tag{15}$$

Let Θ^ψ be the set of $\theta\in\Theta'$ such that ψ_θ is conservative. Obviously, we have:

$$[0,+\infty)\subset\Theta^\psi\subset\Theta'\subset\left(\Theta^\psi\cup\{\theta_\infty\}\right).$$

Let θ^* be the unique positive root of ψ' if it exists. Notice that $\theta^*=0$ if ψ is critical and that θ^* exists and is positive if ψ is super-critical. If θ^* exists, then the branching mechanism ψ_{θ^*} is critical. We set Θ_*^ψ for $[\theta^*,+\infty)$ if θ^* exists and $\Theta_*^\psi=\Theta^\psi$ otherwise. The function ψ is a one-to-one mapping from Θ_*^ψ onto $\psi(\Theta_*^\psi)$. We write ψ^{-1} for the inverse of the previous mapping. In particular, if ψ_θ is (sub)critical then we have $\psi^{-1}(\psi(\theta))=\theta$; and if ψ_θ is super-critical then we have $\theta<\theta^*<\psi^{-1}(\psi(\theta))$. We set:

$$q_0 = \psi^{-1}(0). \tag{16}$$

Note that if ψ is super-critical, then $q_0>0$ and, thanks to (10), $\mathbb{N}^\psi[\sigma=+\infty]=\psi^{-1}(0)>0$.

We recall the Girsanov transformation from [2], which sums up the situation for any branching mechanism ψ. Let ψ be a branching mechanism satisfying (H1-3). Let $\theta\in\Theta^\psi$ and $a>0$. We set:

$$M_a^{\psi,\theta} = \exp\left\{\theta\mathcal{Z}_0-\theta\mathcal{Z}_a-\psi(\theta)\int_0^a\mathcal{Z}_s ds\right\}.$$

Recall that $\mathcal{Z}_0=0$ under $\mathbb{N}^\psi$. For any non-negative measurable functional F defined on $\mathbb{T}$, we have for $\theta\in\Theta^\psi$ and $a\geq 0$:

$$\mathbb{E}_r^{\psi_\theta}[F(\mathcal{T}^{(a)})] = \mathbb{E}_r^\psi\left[F(\mathcal{T}^{(a)})M_a^{\psi,\theta}\right] \quad\text{and}\quad \mathbb{N}^{\psi_\theta}[F(\mathcal{T}^{(a)})] = \mathbb{N}^\psi\left[F(\mathcal{T}^{(a)})M_a^{\psi,\theta}\right]. \tag{17}$$

Furthermore, if $\theta\geq\theta^*$, then we have:

$$\mathbb{E}_r^{\psi_\theta}[F(\mathcal{T})] = \mathbb{E}_r^\psi\left[F(\mathcal{T})\,\mathrm{e}^{\theta r-\psi(\theta)\sigma}\,\mathbf{1}_{\{\sigma<+\infty\}}\right], \tag{18}$$

$$\mathbb{N}^{\psi_\theta}[F(\mathcal{T})] = \mathbb{N}^\psi\left[F(\mathcal{T})\,\mathrm{e}^{-\psi(\theta)\sigma}\,\mathbf{1}_{\{\sigma<+\infty\}}\right], \tag{19}$$

$$\mathbf{N}^{\psi_\theta}[F(\mathcal{T})] = \mathbf{N}^\psi\left[F(\mathcal{T})\,\mathrm{e}^{-\psi(\theta)\sigma}\,\mathbf{1}_{\{\sigma<+\infty\}}\right]. \tag{20}$$

We have that under $\mathbb{P}_r^\psi(d\mathcal{T})$, the random measure $\mathcal{N}_0^{\mathcal{T}}(dx, d\mathcal{T}')$, defined by (3) with $a = 0$, is a Poisson point measure on $\{\emptyset\} \times \mathbb{T}$ with intensity $r\delta_\emptyset(dx)\mathbb{N}^\psi[d\mathcal{T}']$. Then, using the first equality in (17) with $F = 1$, we get that for $\theta \geq \theta^*$ and $a > 0$,

$$\mathbb{N}^{\psi_\theta}\left[1 - \exp\left\{\theta \mathcal{Z}_a + \psi(\theta)\int_0^a \mathcal{Z}_s ds\right\}\right] = -\theta. \tag{21}$$

2.9. **Pruning Lévy trees and CRT-valued Processes.** A general pruning of a Lévy tree has been defined in [6]. Under $\mathbb{N}^\psi[d\mathcal{T}]$ and conditionally on $\mathcal{T}$, we consider a mark process $M^{\mathcal{T}}(d\theta, dy)$ on the tree which is a Poisson point measure on $\mathbb{R}_+ \times \mathcal{T}$ with intensity:

$$\mathbf{1}_{[0,+\infty)}(\theta)d\theta\left(2\beta\ell^{\mathcal{T}}(dy) + \sum_{x\in\mathrm{Br}_\infty(\mathcal{T})} \Delta_x\delta_x(dy)\right).$$

The atoms $(\theta_i, y_i)_{i\in I}$ of this measure can be seen as marks that arrive on the tree, y_i being the location of the mark and θ_i the "time" at which it appears. There are two kinds of marks: some are "uniformly" distributed on the skeleton of the tree (they correspond to the term $2\beta\ell^{\mathcal{T}}$ in the intensity) whereas the others lay on the infinite branching points of the tree: an infinite branching point y being first marked after an exponential time with parameter Δ_y.

For every $x \in \mathcal{T}$, we set:

$$\theta(x) = \inf\{\theta > 0,\ M^{\mathcal{T}}([0,\theta] \times [\![\emptyset, x]\!]) > 0\},$$

which is called the record process on the tree as defined in [1]. This corresponds to the first time at which a mark arrives on $[\![\emptyset, x]\!]$. Using this record process, we define the pruned tree at time q as:

$$\mathcal{T}_q = \{x \in \mathcal{T},\ \theta(x) \geq q\}$$

with the induced metric, root $\emptyset$ and mass measure the restriction of the mass measure $\mathbf{m}^{\mathcal{T}}$. If one cuts the tree $\mathcal{T}$ at time θ_i at point y_i, then $\mathcal{T}_q$ corresponds to the resulting sub-tree of $\mathcal{T}$ containing the root at time q. According to [6], Theorem 1.1, for fixed $q > 0$, the distribution of $\mathcal{T}_q$ under $\mathbb{N}^\psi$ is $\mathbb{N}^{\psi_q}$. We set:

$$\sigma_q = \mathbf{m}^{\mathcal{T}_q}(\mathcal{T}_q).$$

Because of the pruning procedure, we have $\mathcal{T}_\theta \subset \mathcal{T}_q$ for $0 \leq q \leq \theta$. The tree-valued process $(\mathcal{T}_q, q \geq 0)$ is a Markov process under $\mathbb{N}^\psi$, see [2]. The process $(\mathcal{T}_q, q \geq 0)$ is a non-increasing process (for the inclusion of trees), and is càdlàg. We recall the transition probabilities for the time reversed process which are given by the so-called special Markov property (see [6] Theorem 4.2 or [2] Theorem 5.6).

Theorem 2.1. *Let ψ be a branching mechanism satisfying (H1-3). Let $0 \leq q \leq \theta$ and $\mathcal{T}_\theta$ distributed according to $\mathbb{N}^{\psi_\theta}$. Conditionally on $\mathcal{T}_\theta$, let $\sum_{i\in I^{\theta,q}} \delta_{(x_i, \mathcal{T}_q^i)}$ be a Poisson point measure on $\mathcal{T}_\theta \times \mathbb{T}$ with intensity:*

$$\mathbf{m}^{\mathcal{T}_\theta}(dx)\mathcal{N}_{\theta-q}^{\psi_q}[d\mathcal{T}].$$

Then, under $\mathbb{N}^\psi$, $(\mathcal{T}_\theta, \mathcal{T}_q)$ is distributed as:

$$\left(\mathcal{T}_\theta,\ \mathcal{T}_\theta \circledast_{i\in I^{\theta,q}} (\mathcal{T}_q^i, x_i)\right).$$

According to (15), the intensity $\mathcal{N}_{\theta-q}^{\psi_q}$ is given by (11) with ψ replaced by ψ_q and $\pi(dr)$ replaced by $\mathrm{e}^{-qr}\,\pi(dr)$, that is:

$$\mathcal{N}_{\theta-q}^{\psi_q}[d\mathcal{T}] = 2\beta(\theta - q)\mathbb{N}^{\psi_q}[d\mathcal{T}] + \int_{(0,+\infty)} \mathrm{e}^{-qr}\,\pi(dr)(1 - \mathrm{e}^{-(\theta-q)r})\mathbb{P}_r^{\psi_r}(d\mathcal{T}). \tag{22}$$

The time-reversed process is a Markov process and its infinitesimal transitions are described in [4].

3. Sub-tree Processes

3.1. **Sub-tree of the Lévy tree.** Following [14], we define a sub-tree process obtained from pruned CRTs and Poissonian selection of leaves. Let ψ be a branching mechanism satisfying (H1-3). We set:

$$\eta = \psi^{-1}(\lambda) \quad \text{for} \quad \lambda \geq 0. \tag{23}$$

Notice that $\psi(\lambda) = \eta$ and, with q_0 defined by (16), $\eta > q_0$ if $\lambda > 0$.

Conditionally on the tree-valued process $(\mathcal{T}_\theta, \theta \in \Theta^\psi)$, we consider the point measure of marked leaves of $\mathcal{T}_0$:

$$\mathcal{P}^0(dt, dx) = \sum_{i \in I_0} \delta_{(t_i, x_i)} \tag{24}$$

defined as a Poisson point measure on $\mathbb{R}_+ \times \mathcal{T}_0$ with intensity measure $dt\, \mathbf{m}^{\mathcal{T}_0}(dx)$. We set:

$$M_\lambda = \mathcal{P}^0([0, \lambda] \times \mathcal{T}_0)$$

the number of marked leaves in $\mathcal{T}_0$. We shall be working on $\{M_\lambda \geq 1\}$ and consider the probability measure:

$$\mathbb{P}^{\psi,\lambda}(d\mathcal{T}) = \mathbb{N}^\psi[d\mathcal{T} \,|\, M_\lambda \geq 1]. \tag{25}$$

Notice that $\eta = \mathbb{N}^\psi[M_\lambda \geq 1]$. We might write $\mathcal{P}^\theta(dt, dx) = \sum_{i \in I_\theta} \delta_{(t_i, x_i)}(dt, dx)$ for the restriction of $\mathcal{P}^0$ to $\mathbb{R}_+ \times \mathcal{T}_\theta$ for $\theta \geq 0$. On $\{M_\lambda \geq 1\}$, for $\theta \geq 0$, we define the pruned sub-tree $\tau_\theta(\lambda)$ containing the root and all the ancestors in $\mathcal{T}_\theta$ of the marked leaves of $\mathcal{T}_0$:

$$\tau_0(\lambda) = \bigcup_{i \in I_0, t_i \leq \lambda} [\![\emptyset, x_i]\!] \quad \text{and} \quad \tau_\theta(\lambda) = \tau_0(\lambda) \bigcap \mathcal{T}_\theta \tag{26}$$

if $\lambda > 0$, and if $\lambda = 0$, we set:

$$\tau_\theta(0) = \bigcap_{\lambda > 0} \tau_\theta(\lambda).$$

Notice that $\tau_\theta(0) = 0$ if $\mathcal{T}_0$ has finite mass measure, whereas $\tau_\theta(0) \neq \emptyset$ (and $\tau_\theta(0)$ has no leaf) if $\mathcal{T}_0$ has infinite mass. By construction, we have a.s. that $\tau_\theta(\lambda)$ is compact if and only if $\mathcal{T}_\theta$ is compact (that is $\mathcal{T}_\theta$ has finite mass measure). The sub-tree $\tau_\theta(\lambda)$ of $\mathcal{T}_\theta$ and thus of $\mathcal{T}_0$ is endowed with the obvious metric. We shall consider the following mass measure on $\tau_\theta(\lambda)$:

$$\mathbf{m}^{\tau_\theta(\lambda)} = \frac{1}{\psi_\theta(\eta)} \sum_{x \in \mathrm{Lf}(\tau_\theta(\lambda))} \delta_x. \tag{27}$$

As θ varies, we obtain a sub-tree process with parameter λ: $\tau(\lambda) = (\tau_\theta(\lambda), \theta \geq 0)$ which is a non-decreasing tree-valued stochastic process, that is for $q < \theta$, $\tau_\theta(\lambda) \subset \tau_q(\lambda)$.

3.2. **Reconstruction of the Lévy tree.** Let g be a generating function of a distribution $p = (p(n), n \in \mathbb{N})$ such that $g'(0) = 0$ (i.e. $p(1) = 0$) and let $c > 0$. We shall define by recursion a Galton-Watson real tree with reproduction distribution p and branch length distributed according to an exponential random variable with mean $1/c$.

Recall Notation (2) for the grafting procedure of trees without mass measure. We say that a discrete rooted real tree $\mathcal{G}$ is a (g, c)-Galton-Watson real tree if $\mathcal{G}$ is distributed as:

$$[\![\emptyset, x]\!] \tilde{\circledast}_{1 \leq k \leq K} (\mathcal{G}_k, x),$$

with:

- $[\![\emptyset, x]\!]$ a real tree rooted at $\emptyset$ with no branching point and such that $E_\emptyset = d(\emptyset, x)$ is a random exponential variable with parameter c,
- K has generating function g and is independent of $E_\emptyset$,
- $(\mathcal{G}_k, k \in \mathbb{N}^*)$ is a sequence of independent rooted real trees which have the same distribution as $\mathcal{G}$ and are independent of $E_\emptyset$ and K.

Let $\lambda \geq 0$ and $\eta = \psi^{-1}(\lambda)$ such that $\eta > 0$. We consider the following generating function:

$$g_{(\psi,\lambda)}(r) = r + \frac{\psi((1-r)\psi^{-1}(\lambda))}{\psi^{-1}(\lambda)\psi'(\psi^{-1}(\lambda))} = r + \frac{\psi((1-r)\eta)}{\eta\psi'(\eta)}. \tag{28}$$

Notice that:

$$g'_{(\psi,\lambda)}(0) = 0 \quad \text{and} \quad g'_{(\psi,\lambda)}(1) = 1 - \frac{\psi'(0)}{\psi'(\eta)}. \tag{29}$$

We write $\mathcal{G}(\psi, \lambda)$ for the $(g_{\psi,\lambda}, \psi'(\eta))$-Galton-Watson real tree. According to Theorem 3.2.1 in [12], if ψ is (sub)critical, then the discrete tree $\tau_0(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$ is distributed as a Galton-Watson tree $\mathcal{G}(\psi, \lambda)$ with mass measure given by (27). Furthermore, we can reconstruct the Lévy tree $\mathcal{T}$ from $\tau_0(\lambda)$, thanks to [14]. For this, recall Definition (12) of $\mathbf{N}$ and define the following probability measure on $\mathbb{R}_+$:

$$\Gamma^{\psi}_{d,\lambda}(dr) = \mathbf{1}_{\{d=2\}} \frac{2\beta}{\psi''(\eta)} \delta_0(dr) + \frac{r^d \,\mathrm{e}^{-r\eta}}{|\psi^{(d)}(\eta)|} \pi(dr). \tag{30}$$

Theorem 3.1 (Theorem 5.6 of [14])**.** *Assume that ψ is (sub)critical and (H1-3) hold. Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. Under $\mathbb{P}^{\psi,\lambda}$ and conditionally on $\tau_0(\lambda)$, $\mathcal{T}_0$ is distributed as:*

$$\tilde{\tau}_0(\lambda) \circledast_{i\in I} (\mathcal{T}_i, x_i) \circledast_{x\in \mathrm{Br}(\tau_0(\lambda))} (\mathcal{T}'_x, x),$$

with:

- *$\tilde{\tau}_0(\lambda)$ as $\tau_0(\lambda)$ but with 0 as mass measure,*
- *$\sum_{i\in I} \delta_{(x_i,\mathcal{T}_i)}$ is a Poisson point measure on $\mathcal{T}_0(\lambda)\times\mathbb{T}$ with intensity $\ell^{\tau_0(\lambda)}(dx)\, \mathbf{N}^{\psi_\eta}[d\mathcal{T}]$,*
- *conditionally on $\sum_{i\in I} \delta_{(x_i,\mathcal{T}_i)}$, the trees $(\mathcal{T}'_x, x \in \mathrm{Br}(\tau_0(\lambda)))$ are independent with $\mathcal{T}'_x$ is distributed as*

$$\int \Gamma^{\psi}_{\kappa(x),\lambda}(dr)\, \mathbb{P}^{\psi_\eta}_r[d\mathcal{T}].$$

Remark 3.2. In fact, in Theorem 5.6 of [14], ψ can be super-critical and $\lambda \geq 0$ with $\eta = \psi^{-1}(\lambda) > 0$. But it is not obvious that in this case the super-critical Lévy tree distribution defined in [14] and the super-critical Lévy tree distribution defined in [2] and recalled here in Section 2.6, are in fact the same. However, we deduce from Remark 5.2 that this equality indeed holds.

4. Marginal distributions

The main goal of this section is to study the one-dimensional distribution of the sub-tree process $\tau(\lambda) = (\tau_\theta(\lambda), \theta \geq 0)$.

We first give an application of the special Markov property.

Proposition 4.1. *Let ψ be a branching mechanism satisfying (H1-3). Let $\lambda \geq 0$ and $\eta = \psi^{-1}(\lambda)$. Under $\mathbb{N}^\psi$, the couple of trees $(\mathcal{T}_\theta, \tau_\theta(\lambda))$ on $\{M_\lambda \geq 1\}$ is distributed as $(\mathcal{T}_0, \tau_0(\psi_\theta(\eta)))$ under $\mathbb{N}^{\psi_\theta}$ on $\{M_{\psi_\theta(\eta)} \geq 1\}$.*

Proof. We first assume $\lambda > 0$. From the special Markov property of Theorem 2.1 for the process $(\mathcal{T}(\theta), \theta \geq 0)$ under $\mathbb{N}^\psi$, we get:

$$\mathcal{T}_0 = \mathcal{T}_\theta \circledast_{j\in J^{\theta,0}} (\mathcal{T}_0^j, y_j), \tag{31}$$

where $\sum_{j\in J^{\theta,0}} \delta_{(y_j,\mathcal{T}_0^j)}$ is, conditionally on $\mathcal{T}_\theta$, a Poisson point measure on $\mathcal{T}_\theta \times \mathbb{T}$ with intensity $\mathbf{m}^{\mathcal{T}_\theta}(dy)\mathcal{N}_\theta^\psi[d\mathcal{T}]$.

Recall $\mathcal{P}^\theta$ is the restriction to $\mathbb{R}_+ \times \mathcal{T}_\theta$ of $\mathcal{P}^0$ defined by (24), and thus $\mathcal{P}_1^\theta = \sum_{i\in I_\theta} \delta_{x_i} \mathbf{1}_{\{t_i\leq\lambda\}}$ is a Poisson point measure on $\mathcal{T}_\theta$ with intensity $\lambda \mathbf{m}^{\mathcal{T}_\theta}(dx)$.

For $j \in J^{\theta,0}$, let $s_j = \inf\{t_i; x_i \in \mathcal{T}_0^j \text{ for } i \in I_0\}$. Notice that conditionally on $\mathcal{T}_0^j$, s_j has an exponential distribution with parameter $\lambda \mathbf{m}^{\mathcal{T}_0}(\mathcal{T}_0^j)$. We deduce that, conditionally on $\mathcal{T}_\theta$, $\mathcal{P}_2^\theta = \sum_{j\in J^{\theta,0}} \delta_{y_j} \mathbf{1}_{\{s_j\leq\lambda\}}$ is a Poisson point measure on $\mathcal{T}_\theta \times \mathbb{R}_+$ with intensity:

$$\mathbf{m}^{\mathcal{T}_\theta}(dx)\mathcal{N}_\theta^\psi \left[1 - \mathrm{e}^{-\lambda\sigma}\right] = [\psi(\theta + \eta) - \psi(\theta) - \psi(\eta)]\ \mathbf{m}^{\mathcal{T}_\theta}(dx),$$

where we use (13) to get the equality. By construction $\mathcal{P}_1^\theta$ and $\mathcal{P}_2^\theta$ are independent Poisson point measures. Therefore, $\mathcal{P}_1^\theta + \mathcal{P}_2^\theta$ is a Poisson point measure with intensity:

$$\mathbf{m}^{\mathcal{T}_\theta}(dx)\,[\lambda + \psi(\theta + \eta) - \psi(\theta) - \psi(\eta)] = \psi_\theta(\eta)\mathbf{m}^{\mathcal{T}_\theta}(dx).$$

To conclude, notice that $\tau_\theta(\lambda)$ is the sub-tree generated by the marked leaves before time λ of $\mathcal{T}_\theta$, which are given by the atoms of $\mathcal{P}_1^\theta$, and the roots x_j of the trees $\mathcal{T}_0^j$ having marked leaves before time λ, that is the atoms of $\mathcal{P}_2^\theta$. Then use that $\mathcal{T}_\theta$ under $\mathbb{N}^\psi$ is distributed as $\mathcal{T}_0$ under $\mathbb{N}^{\psi_\theta}$ to conclude.

For $\lambda = 0$, we have $\mathcal{P}_1^\theta = 0$ and $\mathcal{T}_0^j$ contributes to $\tau_\theta(0)$ if and only if it has infinite mass. So, in the previous argument, one has to replace $\mathcal{P}_2^\theta$ by $\sum_{j\in J^{\theta,0}} \delta_{y_j} \mathbf{1}_{\{\sigma^{\mathcal{T}_0^j}=+\infty\}}$ which is a Poisson point measure with intensity:

$$\mathbf{m}^{\mathcal{T}_\theta}(dx)\mathcal{N}_\theta^\psi\,[\sigma = +\infty] = \psi_\theta(\eta)\,\mathbf{m}^{\mathcal{T}_\theta}(dx).$$

Hence the conclusion follows. □

Remark 4.2. Assume $\lambda > 0$. Using the notation from the previous proof, for $k \in \mathbb{N}^*$, we let:

$$Y_k = \mathrm{Card}\,\{j \in J^{\theta,0};\ \ \mathrm{Card}\ \left(\mathrm{Lf}(\tau_0(\lambda)) \cap \mathcal{T}_0^j\right) = k\}$$

be the number of trees grafted on $\mathcal{T}_\theta$ having exactly k leaves marked at time λ and $Y_0 = \langle \mathcal{P}_1^\theta, \mathbf{1}\rangle = \mathrm{Card}\ \ (\mathrm{Lf}(\tau_\theta(\lambda)) \cap \mathrm{Lf}(\mathcal{T}))$ be the number of marked leaves on $\mathcal{T}_\theta$. We get that conditionally on $\mathcal{T}_\theta$, the random variables $(Y_k, k \in \mathbb{N})$ are independent, Y_0 is Poisson with parameter $\lambda\sigma_\theta$, and for $k \in \mathbb{N}^*$, Y_k is Poisson with parameter $\sigma_\theta \mathcal{N}_\theta^\psi \left[(\lambda\sigma)^k\,\mathrm{e}^{-\lambda\sigma}\right]/k!$.

Using the Girsanov transformation from Section 2.8, we will give a Girsanov transformation for $\tau(\lambda)$.

For $\mathcal{T} \in \mathbb{T}$, let $L(\mathcal{T}) = \mathrm{Card}\,\mathrm{Lf}(\mathcal{T})$ be the number of leaves of the tree $\mathcal{T}$ and

$$L(a, \mathcal{T}) = L(a, \mathcal{T}^{(a)}) = \mathrm{Card}\,\{x \in \mathcal{T}; d^{\mathcal{T}}(\emptyset, x) = a\} \tag{32}$$

be the number of elements of $\mathcal{T}$ at distance a from the root. Note that:

$$\eta = \psi^{-1}(\lambda) = q_0 + \psi_{q_0}^{-1}(\lambda), \quad \text{and} \quad \psi'(\eta) = \psi'(\psi^{-1}(\lambda)) = \psi'_{q_0}(\psi_{q_0}^{-1}(\lambda)). \tag{33}$$

We first state a preliminary Lemma. Let $\mathrm{P}^{\psi,\lambda}(d\mathcal{G})$ denote the distribution of the Galton-Watson tree $\mathcal{G}(\psi, \lambda)$ defined in Section 3.2.

Lemma 4.3. *Let ψ be a branching mechanism satisfying (H1-3). Let $\lambda \geq 0$ and $\eta = \psi^{-1}(\lambda) > 0$. For any non-negative measurable function F on $\mathbb{T}$ and $a \geq 0$, we have:*

$$\mathrm{E}^{\psi,\lambda}[F(\mathcal{G}^{(a)})] = \mathrm{E}^{\psi_{q_0},\lambda}\left[F(\mathcal{G}^{(a)})\left(\frac{\eta}{\eta - q_0}\right)^{L(a,\mathcal{G})-1}\right].$$

Proof. Let $(p_{(\psi,\lambda)}(n), n \in \mathbb{N})$ be the probability measure determined by $g_{(\psi,\lambda)}$ defined by (28). Then $p_{(\psi,\lambda)}(1) = 0$ and for $n \neq 1$, we have:

$$p_{(\psi,\lambda)}(n) = \frac{g^{(n)}_{(\psi,\lambda)}(0)}{n!} = \frac{|\psi^{(n)}(\eta)|\,\eta^{n-1}}{\psi'(\eta)n!}. \tag{34}$$

Thanks to (33), we have $\psi_{q_0}^{-1}(\lambda) = \eta - q_0$ and for $n \geq 0$, $\psi^{(n)}(\eta) = \psi^{(n)}_{q_0}(\eta - q_0)$. Set $u = (\eta - q_0)/\eta$. Then, we have for $n \in \mathbb{N}$:

$$\begin{aligned} p_{(\psi_{q_0},\lambda)}(n) &= \frac{|\psi^{(n)}_{q_0}(\psi_{q_0}^{-1}(\lambda))|\,(\psi_{q_0}^{-1}(\lambda))^{n-1}}{\psi'_{q_0}(\psi_{q_0}^{-1}(\lambda))n!}\mathbf{1}_{\{n\neq 1\}} \\ &= u^{n-1}\frac{|\psi^{(n)}(\eta)|\,\eta^{n-1}}{\psi'(\eta)n!}\mathbf{1}_{\{n\neq 1\}} = u^{n-1}p_{(\psi,\lambda)}(n). \end{aligned} \tag{35}$$

The number of leaves of $\mathcal{G}^{(a)}$ which are leaves of $\mathcal{G}$ is $\mathrm{P}^{\psi_{q_0},\lambda}(d\mathcal{G})$-a.s. given, for fixed a, by $L(\mathcal{G}^{(a)}) - L(a,\mathcal{G})$. Thanks to (33), the individual lifetimes under $\mathrm{P}^{\psi,\lambda}$ and $\mathrm{P}^{\psi_{q_0},\lambda}$ have the same distribution. Recall κ_x is the number of children of x. Therefore, we have:

$$\begin{aligned} \mathrm{E}^{\psi,\lambda}[F(\mathcal{G}^{(a)})] &= \mathrm{E}^{\psi_{q_0},\lambda}\left[F(\mathcal{G}^{(a)})\left(\frac{p_{(\psi,\lambda)}(0)}{p_{(\psi_{q_0},\lambda)}(0)}\right)^{L(\mathcal{G}^{(a)})-L(a,\mathcal{G})}\prod_{x\in\mathrm{Br}(\mathcal{G}^{(a)})}\frac{p_{(\psi,\lambda)}(\kappa_x)}{p_{(\psi_{q_0},\lambda)}(\kappa_x)}\right] \\ &= \mathrm{E}^{\psi_{q_0},\lambda}\left[F(\mathcal{G}^{(a)})u^{L(\mathcal{G}^{(a)})-L(a,\mathcal{G})-\sum_{x\in\mathrm{Br}(\mathcal{G}^{(a)})}(\kappa_x-1)}\right] \\ &= \mathrm{E}^{\psi_{q_0},\lambda}\left[F(\mathcal{G}^{(a)})u^{1-L(a,\mathcal{G})}\right], \end{aligned}$$

where the last equality is a consequence of the following fact for finite discrete trees $\mathcal{G}$:

$$1 + \sum_{x\in\mathrm{Br}(\mathcal{G})}(\kappa_x - 1) = L(\mathcal{G}).$$

□

Recall (32). We shall consider the following whole families of sub-trees and leaves:

$$\tau^{(a)}_{\theta,\lambda} = \{\tau^{(a)}_\theta(z), z \geq \lambda\}, \quad L(a,\tau_{\theta,\lambda}) = L(a,\tau^{(a)}_{\theta,\lambda}) = \{L(a,\tau^{(a)}_\theta(z)), z \geq \lambda\}.$$

We have the following Girsanov theorem.

Theorem 4.4. *Let ψ be a branching mechanism satisfying (H1-3). Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. If ψ is super-critical, then for any non-negative measurable functional H on the Skorokhod space $\mathbb{D}([\lambda,+\infty),\mathbb{T})$, we have:*

$$\mathbb{N}^\psi\left[H(\tau^{(a)}_{0,\lambda})\mathbf{1}_{\{M_\lambda\geq 1\}}\right] = \mathbb{N}^{\psi_{q_0}}\left[\left(\frac{\eta}{\eta - q_0}\right)^{L(a,\tau_0(\lambda))}H(\tau^{(a)}_{0,\lambda})\mathbf{1}_{\{M_\lambda\geq 1\}}\right]. \tag{36}$$

Proof. Recall the random measure $\mathcal{N}_a^{\mathcal{T}}$ defined in (3) is according to the branching property (iv), conditionally on $\mathcal{T}^{(a)}$, a Poisson point measure with intensity $\ell^a(dx)\mathbb{N}^\psi[dT]$. We deduce that, conditionally on $\mathcal{T}^{(a)}$, $L(a,\tau_0(\lambda)) = L(a,\tau_0^{(a)}(\lambda))$ is a Poisson random variable with parameter:

$$\mathbb{N}^\psi\left[1-\mathrm{e}^{-\lambda\sigma}\right]\mathcal{Z}_a = \eta\mathcal{Z}_a.$$

Let $\mathcal{P}^\psi$ be, conditionally on $\mathcal{T}^{(a)}$ and $\tau_0^{(a)}(\lambda)$, a Poisson point measure on $[\lambda,+\infty)$ with intensity $\mathcal{Z}_a\left(\psi^{-1}\right)'(z)\,dz$. We consider the family of random variables:

$$\mathcal{P}_\lambda^\psi = \{\mathcal{P}^\psi([\lambda,z]), z\geq\lambda\}.$$

Using again the branching property (iv), we get that, under $\mathbb{N}^\psi$ and conditionally on $\mathcal{T}^{(a)}$, $L\left(a,\tau_{0,\lambda}\right)$ is distributed as $L\left(a,\tau_0(\lambda)\right)+\mathcal{P}_\lambda^\psi := \{L\left(a,\tau_0(\lambda)\right)+\mathcal{P}^\psi([\lambda,z]), z\geq\lambda\}$. Then notice that the first equality of (33) implies that $\mathcal{P}_\lambda^\psi$ under $\mathbb{N}^\psi\left[\,\cdot\,|\mathcal{T}^{(a)}\right]$ is distributed as $\mathcal{P}_\lambda^{\psi_{q_0}}$ under $\mathbb{N}^{\psi_{q_0}}\left[\,\cdot\,|\mathcal{T}^{(a)}\right]$. We set:

$$F(\mathcal{T}^{(a)}, L(a,\tau_{0,\lambda}^{(a)})) = \mathbb{N}^\psi\left[H(\tau_{0,\lambda}^{(a)})\mathbf{1}_{\{M_\lambda\geq 1\}}\mid\ \mathcal{T}^{(a)}, L(a,\tau_{0,\lambda}^{(a)})\right].$$

We deduce that:

$$\begin{aligned}
\mathbb{N}^\psi\left[H(\tau_{0,\lambda}^{(a)})\mathbf{1}_{\{M_\lambda\geq 1\}}\right] &= \mathbb{N}^\psi\left[F(\mathcal{T}^{(a)}, L(a,\tau_{0,\lambda}^{(a)}))\right]\\
&= \mathbb{N}^\psi\left[F(\mathcal{T}^{(a)}, L(a,\tau_0^{(a)}(\lambda))+\mathcal{P}_\lambda^\psi)\right]\\
&= \mathbb{N}^\psi\left[\sum_{k=0}^\infty F(\mathcal{T}^{(a)}, k+\mathcal{P}_\lambda^\psi)\frac{(\eta\mathcal{Z}_a)^k}{k!}\,\mathrm{e}^{-\eta\mathcal{Z}_a}\right]\\
&= \mathbb{N}^{\psi_{q_0}}\left[\sum_{k=0}^\infty F(\mathcal{T}^{(a)}, k+\mathcal{P}_\lambda^{\psi_{q_0}})\frac{(\eta\mathcal{Z}_a)^k}{k!}\,\mathrm{e}^{-(\eta-q_0)\mathcal{Z}_a}\right],
\end{aligned}$$

where we used the conditional independence of $\mathcal{P}^\psi$ and $\tau_0^{(a)}(\lambda)$ given $\mathcal{T}^{(a)}$ for the third equality, the Girsanov transformation (17) for the last equality (and that $\psi(q_0)=0$). Using $\psi_{q_0}^{-1}(\lambda)=\eta-q_0$, we notice that $L(a,\tau_0(\lambda))$ is under $\mathbb{N}^{\psi_{q_0}}\left[\,\cdot\,|\mathcal{T}^{(a)}\right]$ a Poisson random variable with parameter:

$$\mathbb{N}^{\psi_{q_0}}\left[1-\mathrm{e}^{-\lambda\sigma}\right]\mathcal{Z}_a = (\eta-q_0)\mathcal{Z}_a.$$

Therefore, we obtain:

$$\begin{aligned}
\mathbb{N}^\psi\left[H(\tau_{0,\lambda}^{(a)})\mathbf{1}_{\{M_\lambda\geq 1\}}\right] &= \mathbb{N}^{\psi_{q_0}}\left[\sum_{k=0}^\infty\left(\frac{\eta}{\eta-q_0}\right)^k F(\mathcal{T}^{(a)}, k+\mathcal{P}_\lambda^{\psi_{q_0}})\frac{((\eta-q_0)\mathcal{Z}_a)^k}{k!}\,\mathrm{e}^{-(\eta-q_0)\mathcal{Z}_a}\right]\\
&= \mathbb{N}^{\psi_{q_0}}\left[\left(\frac{\eta}{\eta-q_0}\right)^{L(a,\tau_0(\lambda))} F(\mathcal{T}^{(a)}, L(a,\tau_0(\lambda))+\mathcal{P}_\lambda^{\psi_{q_0}})\right]\\
&= \mathbb{N}^{\psi_{q_0}}\left[\left(\frac{\eta}{\eta-q_0}\right)^{L(a,\tau_0(\lambda))} F(\mathcal{T}^{(a)}, L(a,\tau_{0,\lambda}^{(0)}))\right],
\end{aligned}$$

where we used for the last equality that under $\mathbb{N}^{\psi_{q_0}}$ and conditionally on $\mathcal{T}^{(a)}$, $L\left(a,\tau_{0,\lambda}^{(a)}\right)$ is distributed as $L\left(a,\tau_0(\lambda)\right)+\mathcal{P}_\lambda^{\psi_{q_0}}$.

By construction, the distribution of $\tau_{0,\lambda}^{(a)}$ conditionally on $\mathcal{T}^{(a)}$ and $L(a,\tau_{0,\lambda}^{(a)})$ is the same under $\mathbb{N}^\psi$ and $\mathbb{N}^{\psi_\theta}$ for any $\theta>0$ and in particular for $\theta=q_0$. We deduce (36). □

We immediately deduce the following Corollary.

Corollary 4.5. *Let ψ be a branching mechanism satisfying (H1-3). Let $\lambda>0$ and $\eta=\psi^{-1}(\lambda)$. Under $\mathbb{P}^{\psi,\lambda}$, for each $\theta\geq 0$, the sub-tree $\tau_\theta(\lambda)$ is distributed as the Galton-Watson real tree $\mathcal{G}(\psi_\theta,\psi_\theta(\eta))$ with mass measure given by (27).*

Recall $\mathrm{P}^{\psi,\lambda}(d\mathcal{G})$ denotes the distribution of the Galton-Watson tree $\mathcal{G}(\psi,\lambda)$ defined in Section 3.2.

Proof. If ψ is (sub)critical, then this is a consequence of Theorem 3.2.1 in [12] and Proposition 4.1. Now we assume that ψ is super-critical. Notice that:

$$\frac{\eta}{\eta-q_0}=\frac{\mathbb{N}^\psi[M_\lambda\geq 1]}{\mathbb{N}^{\psi_{q_0}}[M_\lambda\geq 1]}.$$

Using Theorem 4.4, this gives that for $a>0$ and G a non-negative measurable functional defined on $\mathbb{T}$:

$$\mathbb{E}^{\psi,\lambda}\left[G(\tau_0^{(a)}(\lambda))\right]=\mathbb{E}^{\psi_{q_0},\lambda}\left[\left(\frac{\eta}{\eta-q_0}\right)^{L(a,\tau_0(\lambda))-1}G(\tau_0^{(a)}(\lambda))\right].$$

Recall that if $(\mathcal{T},\emptyset,d,\mathbf{m})$ is a measured rooted real tree, then we denote by $\tilde{\mathcal{T}}$ the real tree $(\mathcal{T},\emptyset,d)$. Since ψ_{q_0} is sub-critical, thanks to Theorem 3.2.1 in [12], we get that under $\mathbb{P}^{\psi_{q_0},\lambda}$, $\tilde\tau_0(\lambda)$ has distribution $\mathrm{P}^{\psi_{q_0},\lambda}$. Then by Lemma 4.3, we get that under $\mathbb{P}^{\psi,\lambda}$, $\tilde\tau_0(\lambda)$ has distribution $\mathrm{P}^{\psi,\lambda}$. Then use Proposition 4.1 to get that for each $\theta\geq 0$, $\tilde\tau_\theta(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$ has distribution $\mathrm{P}^{\psi_\theta,\psi_\theta(\eta)}$. □

The following Corollary is another direct consequence of Theorem 4.4.

Corollary 4.6. *Let $\lambda>0$ and $a>0$ be fixed. Under $\mathbb{N}^{\psi_{q_0}}$ on $\{M_\lambda\geq 1\}$, the process $(Q_z,z\geq\lambda)$ defined by:*

$$Q_z=\left(\frac{\psi^{-1}(z)}{\psi^{-1}(z)-q_0}\right)^{L(a,\tau_0(z))}$$

is a backward martingale with respect to the filtration $(\mathcal{Q}_z,z\geq\lambda)$ with $\mathcal{Q}_z=\sigma(\tau_0(z');z'\geq z)$.

We present an other Girsanov transformation for sub-trees.

Remark 4.7. Let ψ be a branching mechanism satisfying (H1-3). For any $q\geq\theta\geq 0$, $a>0$ and F a non-negative measurable functional, we have:

$$\mathbb{E}^{\psi,\lambda}\left[F(\tilde\tau_q^{(a)}(\lambda))\right]=\mathbb{E}^{\psi,\lambda}\left[F(\tilde\tau_\theta^{(a)}(\lambda))N_{a,\lambda}^{\theta,q}(\tau_\theta(\lambda))\right],\tag{37}$$

where $N_\lambda^{\theta,q}$ is defined for discrete trees by:

$$N_{a,\lambda}^{\theta,q}(T)=\left(\frac{\psi_q(\eta)}{\psi_\theta(\eta)}\right)^{L(T^{(a)})-L(a,T)}\mathrm{e}^{(\psi'(\theta+\eta)-\psi'(q+\eta))\ell^T(T^{(a)})}\prod_{x\in\mathrm{Br}(T^{(a)})}\frac{\psi_q^{(\kappa_x)}(\eta)}{\psi_\theta^{(\kappa_x)}(\eta)},$$

with the convention $\prod_{x\in\emptyset}=1$. Under $\mathbb{P}^{\psi,\lambda}$, the process $N_\lambda^{\theta,q}=\left(N_{a,\lambda}^{\theta,q}(\tilde\tau_\theta(\lambda)),a\geq 0\right)$ is a martingale with respect to the filtration $\left(\sigma(\tilde\tau_\theta^{(a)}(\lambda)),a\geq 0\right)$.

5. Convergence of the sub-tree processes

We provide an alternative proof of the convergence of the sub-trees to the Lévy tree from [14] using the Gromov-Hausdorff-Prohorov distance on $\mathbb{T}$ which relies on the Girsanov transformation. Recall that for simplicity, we identify T and $(T, d^T, \emptyset^T, \mathbf{m}^T) \in \mathbb{T}$. And, under $\mathbb{P}_r^\psi$ or $\mathbb{N}^\psi$, the mass measure on $\tau_0(\lambda)$ is given by (27).

Theorem 5.1. *Let ψ be a branching mechanism satisfying (H1-3). We have $\mathbb{N}^\psi$-a.e. or $\mathbb{P}_r^\psi$-a.s.:*

$$\lim_{\lambda\to+\infty} d_{GHP}(\mathcal{T}, \tau_0(\lambda)) = 0. \tag{38}$$

Proof. Under $\mathbb{N}^\psi$, the convergence (38) is a consequence of Lemma 5.4 below (see also Proposition 2.8 in [5] to get the d_{GHP} convergence from the d^c_{GHP} convergence) for the (sub)critical case and Lemma 5.5 below for the super-critical case. Then the $\mathbb{P}_r^\psi$-a.s. convergence is a consequence of the representation of $\mathbb{P}_r^\psi$ from Section 2.7. □

Remark 5.2. Notice in particular that Theorem 5.1 asserts that $(\mathcal{F}, (\mathcal{F}(\lambda), \lambda \geq 0))$ in [14] and $(\mathcal{T}, (\tau_0(\lambda), \lambda \geq 0))$ have the same distribution. In particular, this implies that the distribution for super-critical Lévy trees defined in [14] based on a coloring leaves process and the one defined in [4] based on a Girsanov transformation are the same. Therefore, Theorem 3.1 is also valid for ψ super-critical.

Remark 5.3. The pruning sub-tree process $(\tau_\theta(\lambda), \theta \geq 0)$ is a piece-wise càdlàg $\mathbb{T}$-valued process. It is easy to check, using the representation of the backward process in [4], that the pruning tree process $(\mathcal{T}_\theta, \theta \geq 0)$ is also a càdlàg $\mathbb{T}$-valued process. Then, one can also prove the convergence, with respect to the Skorokhod topology, of the pruning sub-tree process $(\tau_\theta(\lambda), \theta \geq 0)$ towards the pruning tree process $(\mathcal{T}_\theta, \theta \geq 0)$ as λ goes to infinity.

Lemma 5.4 is stated in Section 5.2 and Lemma 5.5 in Section 5.3. Section 5.1 presents preliminaries on approximation of trees by discrete sub-trees.

5.1. **Distance between trees and discrete sub-trees.**

In this Section, we present an immediate convergence result from sub-trees to trees for trees coded by a function.

Let f be a non-negative continuous function with compact support s.t. $f(0) = 0$. We set $\sigma = \sup\{t; f(t) > 0\}$. We define:

$$d^f(x, y) = f(x) + f(y) - 2 \inf_{u\in[x\wedge y, x\vee y]} f(u)$$

and the equivalence relation: $x \sim y$ if $d^f(x, y) = 0$. We set $T^f = [0, \sigma]/\sim$. Let p^f be the projection from $[0, \sigma]$ to T^f, with $p^f(x)$ the equivalent class of x in T^f. Let $\mathbf{m}^f$ be the image of the Lebesgue measure on $[0, \sigma]$ by the projection p^f. Set $\emptyset^f = p^f(0)$ and we still denote by d^f the distance on T, image of d^f by p^f. It is well known that $(T^f, d^f, \emptyset^f, \mathbf{m}^f)$ is a measured rooted compact real tree.

Let $\Delta = \{y_0, \ldots, y_{N_\Delta}\}$, with $1 \leq N_\Delta < +\infty$ and $0 = y_0 < \cdots < y_{N_\Delta} \leq \sigma$, be a finite subdivision of $[0, \sigma]$. Let $|\Delta| = \sup_{0\leq i<N_\Delta} y_{i+1} - y_i$ be the mesh of the subdivision. For $0 \leq i < N_\Delta$, let $\bar{y}_i \in [y_i, y_{i+1}]$ such that $f(\bar{y}_i) = \inf_{u\in[y_i,y_{i+1}]} f(u)$. We consider f_Δ the linear interpolation of the points $\{(y_i, f(y_i)), (\bar{y}_i, f(\bar{y}_i)); 0 \leq i < N_\Delta\} \cup \{(y_{N_\Delta}, f(y_{N_\Delta}))\}$. By construction T^{f_Δ} is the smallest sub-tree of T^f containing $\{p^f(y_i), 0 \leq i \leq N_\Delta\}$.

Let $a_\Delta \geq 0$ and $\mathbf{m}^{f,\Delta}$ be the image of the measure $\mu_\Delta = a_\Delta \sum_{y\in\Delta, y\neq 0} \delta_y$ by the projection p^f. We consider the measured rooted real tree $T^{f,\Delta} = (T^{f_\Delta}, d^{f_\Delta}, \emptyset^f, \mathbf{m}^{f,\Delta})$. It is elementary to get:

$$d^c_{\mathrm{GHP}}(T^f, T^{f,\Delta}) \leq \sup_{|x-y|\leq|\Delta|} |f(x) - f(y)| + d_{\mathrm{P}}^{[0,\sigma]}(\mathrm{Leb}, \mu_\Delta), \tag{39}$$

where Leb is the Lebesgue measure on $[0, \sigma]$, and the space $[0, \sigma]$ is endowed with the usual distance.

5.2. **The (sub)critical case.** The main result of this Section is the following Lemma.

Lemma 5.4. *Let ψ be a (sub)critical branching mechanism satisfying (H1-3). We have $\mathbb{N}^\psi$-a.e. for all $a_0 \geq 0$:*

$$\lim_{\lambda\to+\infty} d^c_{GHP}(\mathcal{T}, \tau_0(\lambda)) = 0 \quad \textit{and} \quad \lim_{\lambda\to+\infty} \sup_{a\leq a_0} d^c_{GHP}(\mathcal{T}^{(a)}, \tau_0^{(a)}(\lambda)) = 0. \tag{40}$$

Proof. According to [12], there exists a continuous stochastic process h, called the height process, such that under its excursion measure it has compact support $[0, \sigma^h]$ and (T^h, σ^h) is distributed at $(\mathcal{T}, \sigma)$ under $\mathbb{N}^\psi$. Notice that the continuity of the height process is a consequence of (H2). Conditionally on h, let $\mathcal{P} = \sum_{i\in I} \delta_{(y_i,t_i)}$ be a Poisson point measure on $[0, \sigma] \times \mathbb{R}_+$ with intensity $dydt$. For $\lambda > 0$, we set:

$$\Delta_\lambda = \{y_i; i \in I \text{ and } t_i \leq \lambda\} \cup \{0\} \quad \text{and} \quad \mu_{\Delta_\lambda} = \frac{1}{\lambda} \sum_{y\in\Delta_\lambda, y\neq 0} \delta_y.$$

By construction, we get the following equality in distribution:

$$(T^h, (T^{h,\Delta_\lambda}, \lambda \geq 0)) \overset{\text{(d)}}{=} (\mathcal{T}, (\tau_0(\lambda), \lambda \geq 0)).$$

The properties of the Poisson point measures imply that a.e. under the excursion measure of h, $\lim_{\lambda\to+\infty} |\Delta_\lambda| = 0$ and $\lim_{\lambda\to+\infty} d_{\mathrm{P}}^{[0,\sigma^h]}(\mathrm{Leb}, \mu_{\Delta_\lambda}) = 0$. Thus, we deduce from Section 5.1 and (39) that a.e. under the excursion measure of h,

$$\lim_{\lambda\to+\infty} d^c_{\mathrm{GHP}}(T^h, T^{h,\Delta_\lambda}) = 0.$$

Thus, we obtain the first part of (40).

We set $\varepsilon_\lambda = d^c_{\mathrm{GHP}}(\mathcal{T}, \tau_0(\lambda))$. According to the proof of Proposition 2.8 in [5], we have, for $a \geq 0$:

$$d^c_{\mathrm{GHP}}(\mathcal{T}^{(a)}, \tau_0^{(a)}(\lambda)) \leq 3\varepsilon_\lambda + \mathbf{m}^{\mathcal{T}} \left(\mathcal{T}^{(a+2\varepsilon_\lambda)} \backslash \mathcal{T}^{(a-\varepsilon_\lambda)} \right).$$

Using (6) and the definition of $\mathcal{Z}$, we deduce that for $a_0 \geq 0$:

$$\sup_{a\leq a_0} d^c_{\mathrm{GHP}}(\mathcal{T}^{(a)}, \tau_0^{(a)}(\lambda)) \leq 3 \left(1 + \sup_{a\leq a_0+2\varepsilon_\lambda} \mathcal{Z}_a \right) \varepsilon_\lambda.$$

We deduce then the second part of (40) from the first part of (40). This ends the proof of the Lemma. $\square$

5.3. **The super-critical case.** The main result of this Section is the following Lemma.

Lemma 5.5. *Let ψ be a super-critical branching mechanism satisfying (H1-3). We have $\mathbb{N}^\psi$-a.e.:*

$$\lim_{\lambda\to+\infty} d_{GHP}(\mathcal{T}, \tau_0(\lambda)) = 0.$$

Proof. We deduce from Theorem 4.4 that for $a > 0$:

$$\mathbb{N}^\psi\left[\liminf_{\lambda\to+\infty}\int_0^a \mathrm{e}^{-r}\left(1\wedge d^c_{\mathrm{GHP}}\left(\mathcal{T}^{(r)}, \tau_0^{(r)}(\lambda)\right)\right)\, dr > 0\right]$$
$$= \mathbb{N}^{\psi_{q_0}}\left[\left(\frac{\psi^{-1}(1)}{\psi^{-1}(1)-q_0}\right)^{L(a,\tau_0(1))}, \liminf_{\lambda\to+\infty}\int_0^a \mathrm{e}^{-r}\left(1\wedge d^c_{\mathrm{GHP}}\left(\mathcal{T}^{(r)}, \tau_0^{(r)}(\lambda)\right)\right)\, dr > 0\right].$$

Then use (40) to get that the right hand-side in the previous equality is 0 for all $a > 0$. This implies that $\mathbb{N}^\psi\left[\liminf_{\lambda\to+\infty} d_{\mathrm{GHP}}(\mathcal{T}, \tau_0(\lambda)) > 0\right] = 0$. □

6. Pruning and growth of the discrete sub-trees

6.1. **The pruning process.** We define the following pruning procedure for the discrete sub-trees. Under $\mathbb{P}^{\psi,\lambda}$, let $\mathfrak{T}$ be distributed as $\tau_0(\lambda)$. Conditionally on $\mathfrak{T}$, we consider a Poisson point measure $\mathcal{M}^{\mathrm{Sk}}(d\theta, dy)$ on $\mathbb{R}_+\times\mathfrak{T}$ with intensity:

$$\psi''(\eta+\theta)\mathbf{1}_{[0,+\infty)}(\theta)d\theta\, \ell^{\mathfrak{T}}(dy)$$

and an independent family of independent random variables $(\xi_x, x\in\mathrm{Br}(\mathfrak{T}))$, such that the distribution of ξ_x has density:

$$-\frac{\psi^{(\kappa_x+1)}(\eta+z)}{\psi^{(\kappa_x)}(\eta)}\mathbf{1}_{\{z>0\}}dz.$$

We define the mark process:

$$\mathcal{M}^{\mathfrak{T}}(d\theta, dy) = \mathcal{M}^{\mathrm{Sk}}(d\theta, dy) + \sum_{x\in\mathrm{Br}(\mathfrak{T})}\delta_{(\xi_x,x)}(d\theta, dy).$$

For every $x\in\mathfrak{T}$, we consider the corresponding record process on $\mathfrak{T}$:

$$\theta^{\mathfrak{T}}(x) = \inf\{\theta>0,\ \mathcal{M}^{\mathfrak{T}}([0,\theta]\times[\![\emptyset, x]\!])>0\}.$$

We define the pruned tree at time $q\geq 0$ as:

$$\mathfrak{T}_q = \{x\in\mathfrak{T},\ \theta^{\mathfrak{T}}(x)\geq q\}$$

with the induced metric, root $\emptyset$ and mass measure $\mathbf{m}^{\mathfrak{T}_q} = \frac{1}{\psi_q(\eta)}\sum_{x\in\mathrm{Lf}(\mathfrak{T}_q)}\delta_x$. Then we have the following theorem.

Theorem 6.1. *Let ψ be a branching mechanism satisfying (H1-3). Let $\lambda\geq 0$ such that $\eta=\psi^{-1}(\lambda)>0$. Then under $\mathbb{P}^{\psi,\lambda}$, the two processes $(\tau_\theta(\lambda), \theta\geq 0)$ and $(\mathfrak{T}_\theta, \theta\geq 0)$ have the same distribution.*

Proof. The proof is based on Theorem 3.1 and Remark 5.2. Notice that the processes $(\tau_\theta(\lambda), \theta\geq 0)$ and $(\mathfrak{T}_\theta, \theta\geq 0)$ are by construction Markov and right continuous. Therefore, it is enough to check the two-dimensional marginals have the same distribution.

Let $\theta\geq q\geq 0$. Recall the pruning procedure defined in Section 2.9. On one hand, a mark appears on the skeleton of $\tau_q(\lambda)$ before time θ, if this is a mark which appears before time θ and which is either on the skeleton of $\mathcal{T}_q$ or on a branching point of $\mathcal{T}_q$. Those marks which initially are on the skeleton of $\mathcal{T}_q$ are distributed on $\tau_q(\lambda)$ according to a Poisson point measure with intensity $2\beta(\theta-q)\ell^{\tau_q(\lambda)}(dy)$. A node of $\mathcal{T}_q$ with mass r has a mark before time θ with probability $1-\mathrm{e}^{-(\theta-q)r}$. And the nodes of $\mathcal{T}_q$ with mass r which lies on the skeleton of $\tau_q(\lambda)$ are, thanks to Theorem 3.1, distributed on $\tau_q(\lambda)$ according to a Poisson point measure with intensity $r\,\mathrm{e}^{-r\eta-rq}\,\pi(dr)\,\ell^{\tau_q(\lambda)}(dy)$. This implies that the marks on the skeleton of $\tau_q(\lambda)$ before time θ are distributed according to a Poisson point measure with intensity:

$$\begin{aligned}\left[2\beta(\theta-q)+\int_{(0,+\infty)}(1-\mathrm{e}^{-(\theta-q)r})r\,\mathrm{e}^{-r\eta-rq}\,\pi(dr)\right]\,\ell^{\tau_q(\lambda)}(dy)&=\left[\psi'_\theta(\eta)-\psi'_q(\eta)\right]\,\ell^{\tau_q(\lambda)}(dy)\\&=\int_q^\theta\psi''(\eta+z)dz\,\ell^{\tau_q(\lambda)}(dy).\end{aligned}$$

On the other hand, if x is a node of $\tau_q(\lambda)$ with number of children κ_x, then a mark appears on it before time θ, if it appears before time θ on $\mathcal{T}_q$. According to Proposition 4.1, $(\mathcal{T}_q,\tau_q(\psi_q(\eta)))$ is distributed as $(\mathcal{T}_0,\tau_0(\lambda))$ under $\mathbb{N}^{\psi_q}$. We deduce, thanks to Theorem 3.1, that the mass Δ_x is conditionally on $\tau_0(\lambda)$ distributed according to $\Gamma^{\psi_q}_{\kappa_x,\psi_q(\eta)}$ defined by (30). Therefore a mark appears on the node x of $\tau_q(\lambda)$ before time θ with probability:

$$\int\Gamma^{\psi_q}_{\kappa_x,\psi_q(\eta)}(dr)(1-\mathrm{e}^{-(\theta-q)r})=1-\frac{\psi_\theta^{(\kappa_x)}(\eta)}{\psi_q^{(\kappa_x)}(\eta)}=\mathbb{P}(\xi_x<\theta|\xi_x>q).$$

By construction of $\mathfrak{T}_\theta$ from $\mathfrak{T}_q$, we deduce that the distribution of $\mathfrak{T}_\theta$ conditionally on $\{\mathfrak{T}_q=T\}$ is the same as the distribution of $\tau_\theta(\lambda)$ conditionally on $\{\tau_q(\lambda)=T\}$. Then use that $\mathfrak{T}_0$ is distributed as $\tau_0(\lambda)$, to deduce that $\mathfrak{T}_\theta$ has the same distribution as $\tau_\theta(\lambda)$. Thus, we get that the processes $(\tau_\theta(\lambda),\theta\geq0)$ and $(\mathfrak{T}_\theta,\theta\geq0)$ have the same two-dimensional marginals distribution. □

Remark 6.2. By construction of $\mathfrak{T}$ and thanks to Proposition 4.1, we get that $(\mathfrak{T}_{\theta+q},\theta\geq0)$ under $\mathbb{P}^{\psi,\lambda}$ is distributed as $(\mathfrak{T}_\theta,\theta\geq0)$ under $\mathbb{P}^{\psi_q,\psi_q(\eta)}$.

6.2. **The growth process.** Let $\lambda>0$. Theorem 6.1 gives the pruning procedure of the sub-tree process. Conversely, we will also give a growth procedure for the time reversed sub-tree process. However, if $\mathcal{T}_\theta$ can be defined on Θ^ψ simultaneously, this is no more the case for $\tau_\theta(\lambda)$. Recall $\psi_\theta(\eta)\geq\lambda>0$ for $\theta\geq0$. We define:

$$\theta_\lambda=\inf\{\theta\in\Theta^\psi;\psi_\theta(\eta)\geq0\}\quad\text{and}\quad\Theta^{\psi,\lambda}=[\theta_\lambda,+\infty)\cap\Theta^\psi. \tag{41}$$

Notice that $\theta_\lambda\leq0$. Theorem 6.1, Remark 6.2 and the Kolmogorov extension theorem insure that there exists a process $(\tau_\theta(\lambda),\theta\in\Theta^{\psi,\lambda})$ under $\mathbb{P}^{\psi,\lambda}$, such that for all $q\in\Theta^{\psi,\lambda}$ the process $(\tau_{\theta+q}(\lambda),\theta\geq0)$ is distributed as $(\tau_\theta(\psi_q(\eta)),\theta\geq0)$ under $\mathbb{P}^{\psi_q,\psi_q(\eta)}$.

We consider the function $g^{q,\theta}_{(\psi,\lambda)}$ defined for $q\in\Theta^{\psi,\lambda}$ and $\theta>q$ by:

$$g^{q,\theta}_{(\psi,\lambda)}(r)=1-\frac{\psi_\theta(\eta(1-r))-\psi_q(\eta(1-r))}{\psi_\theta(\eta)}. \tag{42}$$

Notice that $\psi_q(\eta)>0$ and thus $g^{q,\theta}_{(\psi,\lambda)}(1)=1$, $g^{q,\theta}_{(\psi,\lambda)}(0)=\psi_q(\eta)/\psi_\theta(\eta)$ and for $k\in\mathbb{N}^*$:

$$\left(g^{q,\theta}_{(\psi,\lambda)}\right)^{(k)}(0)=\frac{(-1)^{k+1}\eta^k}{\psi_\theta(\eta)}(\psi^{(k)}(\theta+\eta)-\psi^{(k)}(q+\eta))\geq0.$$

Since ψ is analytical at least on $(\theta_\lambda, +\infty)$, we deduce that $g^{q,\theta}_{(\psi,\lambda)}(r)$ is the generating function of a random variable K taking values in $\mathbb{N}$. Let $(\tau^k, k \in \mathbb{N}^*)$ be independents random trees distributed as $\tau_q(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$ and independent of K. We set:

$$\mathcal{G}_{q,\theta}(\psi,\lambda) = \emptyset \circledast_{1\leq k\leq K} (\tau^k, \emptyset),$$

with the convention that $\emptyset \circledast_{1\leq k\leq K} (\tau^k, \emptyset) = \emptyset$ if $K = 0$.

Theorem 6.3. *Let ψ be a branching mechanism satisfying (H1-3). Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. Let $\theta > q$ with $q \in \Theta^{\psi,\lambda}$. Then under $\mathbb{P}^{\psi,\lambda}$, conditionally on $\tau_\theta(\lambda)$, $\tau_q(\lambda)$ is distributed as*

$$\tau_\theta(\lambda) \circledast_{x\in \mathrm{Lf}(\tau_\theta(\lambda))} (\mathcal{G}^x_q, x),$$

with mass measure given by (27) (with θ replaced by q) and where $(\mathcal{G}^x_q, x \in \mathrm{Lf}(\tau_\theta(\lambda)))$ are independent and distributed according to $\mathcal{G}_{q,\theta}(\psi,\lambda)$.

We first state a preliminary Lemma.

Lemma 6.4. *Under the Hypothesis of Theorem 6.3, the sub-tree $\tilde{\tau}_0(\psi_q(\eta))$ is distributed under $\mathcal{N}^{\psi_q}_{\theta-q}[\,\cdot\,|M_{\psi_q(\eta)} \geq 1]$ as $\mathcal{G}_{q,\theta}(\psi,\lambda)$ conditionally on $\mathcal{G}_{q,\theta}(\psi,\lambda) \neq \emptyset$.*

Proof. By construction of $\mathcal{G}_{q,\theta}(\psi,\lambda)$, the Lemma will be proved as soon as we check that the degree of the root of $\tau_0(\psi_q(\eta))$ under $\mathcal{N}^{\psi_q}_{\theta-q}\left[\,\cdot\,|\, M_{\psi_q(\eta)} \geq 1\right]$ is distributed as K conditionally on $\{K \geq 1\}$.

Without loss of generality, we may write ψ, λ and θ for ψ_q, $\psi_q(\eta)$ and $\theta - q$, that is assume $q = 0$. Let $N_\emptyset$ be the degree of the root $\emptyset$ in $\tau_0(\lambda)$. Notice that $\{M_\lambda \geq 1\} = \{N_\emptyset \geq 1\}$. We set $h(u) = \mathcal{N}^\psi_\theta\left[u^{N_\emptyset}\mathbf{1}_{\{N_\emptyset\geq 1\}}\right]$. Notice that, under $\mathbb{N}^\psi$, $N_\emptyset$ is 0 or 1 and that, under $\mathbb{P}^\psi_r$, $N_\emptyset$ is a Poisson random variable with mean $r\mathbb{N}^\psi[M_\lambda \geq 1] = r\eta$. We deduce that for $u \in [0,1]$:

$$\begin{aligned} h(u) &= 2\beta\theta u \mathbb{N}^\psi[M_\lambda \geq 1] + \int_{(0,+\infty)} \pi(dr)(1 - \mathrm{e}^{-\theta r})\mathbb{E}^\psi_r\left[u^{N_\emptyset}\mathbf{1}_{\{N_\emptyset\geq 1\}}\right] \\ &= 2\beta\theta\eta u + \int_{(0,+\infty)} \pi(dr)(1 - \mathrm{e}^{-\theta r})(\mathrm{e}^{-r\eta(1-u)} - \mathrm{e}^{-r\eta}). \end{aligned}$$

Let $g_0 = g^{0,\theta}_{(\psi,\lambda)}$ be the generating function of K and g_1 be the generating function of K conditionally on $\{K \geq 1\}$. Elementary computations yields $g_0(u) = g_0(0) + h(u)/\psi_\theta(\eta)$. We deduce that $g_1(u) = h(u)/h(1)$. This readily implies that $N_\emptyset$ under $\mathcal{N}^\psi_\theta[\,\cdot\,|\,M_\lambda \geq 1]$ is distributed as K conditionally on $\{K \geq 1\}$. $\square$

Proof of Theorem 6.3. The proof is very similar to the proof of Proposition 4.1. From the special Markov property Theorem 2.1, we get:

$$\mathcal{T}_q = \mathcal{T}_\theta \circledast_{j\in J^{\theta,q}} (\mathcal{T}^j_q, x_j),$$

where $\sum_{j\in J^{\theta,q}} \delta_{(x_j,\mathcal{T}^j_q)}$ is, conditionally on $\mathcal{T}_\theta$, a Poisson point measure on $\mathcal{T}_\theta \times \mathbb{T}$ with intensity $\mathbf{m}^{\mathcal{T}_\theta}(dx)\mathcal{N}^{\psi_q}_{\theta-q}[d\mathcal{T}]$. Notice that $\mathcal{T}^j_q$ gives a contribution to $\tau_q(\lambda)$ (that is $\mathcal{T}^j_q \cap \tau_q(\lambda) \neq \emptyset$) if there is at least one marked leaf on $\mathcal{T}^j_q$. Furthermore, if there is a contribution, then $\mathcal{T}^j_q \cap \tau_q(\lambda)$ is distributed as $\tau_0(\psi_q(\eta))$ under $\mathcal{N}^{\psi_q}_{\theta-q}[\,\cdot\,|M_{\psi_q(\eta)} \geq 1]$ but for the root which is x_j. This distribution is given in Lemma 6.4. Thanks to (13), we have:

$$\mathcal{N}^{\psi_q}_{\theta-q}[M_{\psi_q(\eta)} \geq 1] = \mathcal{N}^{\psi_q}_{\theta-q}[1 - \mathrm{e}^{-\psi_q(\eta)\sigma}] = \psi(\theta+\eta) - \psi(\theta) - \psi_q(\eta) = \psi_\theta(\eta) - \psi_q(\eta). \tag{43}$$

Standard results on marked Poisson point process imply that the point measure on the leaves of $\tau_q(\lambda)$ which are still in $\tau_\theta(\lambda)$, that is $\sum_{x\in \mathrm{Lf}(\tau_\theta(\lambda))\cap \mathrm{Lf}(\tau_q(\lambda))} \delta_x(dy)$, is, conditionally on $\mathcal{T}_\theta$, a Poisson point process on $\mathcal{T}_\theta$ with intensity $\psi_q(\eta)\mathbf{m}^{\mathcal{T}_\theta}(dy)$, and is also independent of $\sum_{j\in J^{\theta,q}} \delta_{(x_j,\mathcal{T}_q^j)}$.

Using standard results on marked Poisson point measure, we get that $\tau_q(\lambda)$ can be recovered from $\tau_\theta(\lambda)$ by grafting independently on each leaf $x \in \mathrm{Lf}(\tau_\theta(\lambda))$:

- Nothing with probability $\psi_q(\eta)/\psi_\theta(\eta)$.
- A sub-tree distributed as $\mathcal{G}_{q,\theta}(\psi,\lambda)$ conditionally on $\mathcal{G}_{q,\theta}(\psi,\lambda) \neq \emptyset$ with probability $1-\psi_q(\eta)/\psi_\theta(\eta)$.

Then use that $\mathrm{P}(\mathcal{G}_{q,\theta}(\psi,\lambda)=\emptyset) = \mathrm{P}(K=0) = \psi_q(\eta)/\psi_\theta(\eta)$ and that the mass measure of $\tau_q(\lambda)$ is given by (27) (with θ replaced by q) to end the proof. □

Remark 6.5. We deduce from Theorem 6.3 that the transition rate (for the backward process) at time θ from $\tau_\theta(\lambda)$ to $\tau_\theta(\lambda) \circledast_{1\le k\le k_0} (\tau^k, x)$, with x a leaf of $\tau_\theta(\lambda)$, is given by:

$$\frac{(-1)^{k_0+1}\eta^{k_0}}{k_0!}\,\frac{\psi^{(k_0+1)}(\theta+\eta)}{\psi_\theta(\eta)}\mu_\theta(d\tau^1)\cdots\mu_\theta(d\tau^{k_0}),$$

with μ_θ the distribution of $\tau_\theta(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$. The mass measure process is always defined by (27).

7. Study of Leaves

We first present a martingale based on the total mass of the pruned process.

Proposition 7.1. *Let ψ be a branching mechanism satisfying (H1-3). Then under $\mathbb{P}_r^\psi$ and $\mathbb{N}^\psi$, the process $(R_\theta, \theta > q_0)$, with:*

$$R_\theta = \psi'(\theta)\sigma_\theta,$$

is a backward martingale with respect to the filtration $(\mathcal{F}_\theta, \theta > q_0)$ where $\mathcal{F}_\theta = \sigma(\mathcal{T}_q, q \ge \theta)$.

Proof. Let $q_0 < q \le \theta$. According to the special Markov property, we have:

$$(\mathcal{T}_\theta, \sigma_q) \overset{(d)}{=} (\mathcal{T}_\theta, \sigma_\theta + \sum_{i\in I} \sigma^{\mathcal{T}^i}),$$

where $\sum_{i\in I} \delta_{\mathcal{T}^i}$ is conditionally on $\mathcal{T}_\theta$ a Poisson point measure on $\mathbb{T}$ with intensity:

$$\mathbf{m}^{\mathcal{T}_\theta}(dx)\mathcal{N}^{\psi_q}_{\theta-q}[d\mathcal{T}].$$

Using (22) and (13), we have:

$$\mathbb{E}_r^\psi[\sigma_q|\mathcal{F}_\theta] = \mathbb{E}_r^\psi\left[\sigma_\theta + \sum_{i\in I}\sigma^{\mathcal{T}^i}\middle|\mathcal{F}_\theta\right] = \sigma_\theta + \sigma_\theta \mathcal{N}^{\psi_q}_{\theta-q}[\sigma] = \frac{\psi'(\theta)}{\psi'(q)}\sigma_\theta.$$

This gives the result under $\mathbb{P}_r^\psi$. The proof is similar under $\mathbb{N}^\psi$. □

Notice that Proposition 7.1 is also a direct consequence of the infinitesimal transitions of time-reversed process $(\mathcal{T}_\theta, \theta\in\Theta^\psi)$ given in [4].

Now we present a result on the number of leaves for the sub-tree process. Let $\lambda \ge 0$. We consider the leaves process of the sub-trees $L(\lambda) = \{L_\theta(\lambda), \theta \in \Theta^{\psi,\lambda}\}$:

$$L_\theta(\lambda) = L(\tau_\theta(\lambda)) = \mathrm{Card}\,(\mathrm{Lf}(\tau_\theta(\lambda))).$$

Proposition 7.2. *Let ψ be a branching mechanism satisfying (H1-3). Let $\lambda \geq 0$ and $\eta = \psi^{-1}(\lambda) > 0$. Under $\mathbb{P}^{\psi,\lambda}$, the process $(R_\theta(\lambda), \theta > q_0)$ with:*

$$R_\theta(\lambda) = \frac{\psi'(\theta)}{\psi_\theta(\eta)} L_\theta(\lambda),$$

is a backward martingale with respect to the filtration $(\mathcal{H}_\theta, \theta > q_0)$, where $\mathcal{H}_\theta = \sigma(\tau_q(\lambda), q \geq \theta)$.

Remark 7.3. Notice that $L_\theta(\lambda)/\psi_\theta(\eta)$ is the total mass of $\mathbf{m}^{\tau_\theta(\lambda)}$. The convergence from Remark 5.3 and the fact that $\mathcal{T}_\theta$ is compact for $\theta > q_0$, implies that (as a process) the total mass of $\mathbf{m}^{\tau_\theta(\lambda)}$ converges to the total mass of $\mathbf{m}^{\mathcal{T}_\theta}$ that is σ_θ as λ goes to infinity. Thus Proposition 7.1 appears as a consequence of Proposition 7.2.

Recall (28) and (42). For $\theta \geq q$ and $q \in \Theta^{\psi,\lambda}$, we set:

$$g_q(r) = g_{(\psi_q, \psi_q(\eta))}(r) \quad \text{and} \quad g(r) = g^{q,\theta}_{(\psi,\lambda)}(r). \tag{44}$$

Proof of Proposition 7.2. We write L_θ for $L_\theta(\lambda)$. Let $q_0 < q \leq \theta$. By Theorem 6.3, we have:

$$\mathbb{E}^{\psi,\lambda}\left[L_q | \tau_\theta(\lambda)\right] = L_\theta g(0) + L_\theta g'(1) \mathbb{E}^{\psi,\lambda}\left[L_q\right]. \tag{45}$$

Thanks to Corollary 4.5 and the branching property, we have:

$$\mathbb{E}^{\psi,\lambda}\left[L_q\right] = g_q(0) + \mathbb{E}^{\psi,\lambda}\left[L_q\right] g_q'(1). \tag{46}$$

This gives:

$$\mathbb{P}^{\psi,\lambda}\left[L_q\right] = \frac{g_q(0)}{1 - g_q'(1)} = \frac{\psi_q(\eta)}{\eta \psi'(q)}.$$

Then use that:

$$g(0) = \frac{\psi_q(\eta)}{\psi_\theta(\eta)}, \quad g'(1) = \frac{\eta}{\psi_\theta(\eta)}\left(\psi'(\theta) - \psi'(q)\right),$$

and (45) to get that:

$$\mathbb{E}^{\psi,\lambda}\left[L_q | \tau_\theta(\lambda)\right] = \frac{\psi_q(\eta)}{\psi'(q)} \frac{\psi'(\theta)}{\psi_\theta(\eta)} L_\theta.$$

This gives the result. □

Remark 7.4. A similar result for the leaves process of discrete time Galton-Watson tree-valued process was proved in Corollary 3.4 of [3] using a quantity similar to $(1 - g_q'(1))/g_q(0)$ which comes from (46).

For $\theta > \theta_\lambda$, the function g_θ is convex positive with $g_\theta(0) > 0$ and $g_\theta(1) = 1$. Hence, for $\zeta \in [0, 1)$, the equation:

$$x = g_\theta(x) + g_\theta(0)(\zeta - 1)$$

has a unique solution $x \in [0, 1]$, which we denote by $h_\theta(\zeta)$. By construction the backward process $(L_\theta(\lambda), \theta > \theta_\lambda)$ is Markov under $\mathbb{P}^{\psi,\lambda}$. The next Proposition gives its one and two-dimensional marginals.

Proposition 7.5. *Let ψ be a branching mechanism satisfying (H1-3). For $\theta \geq q > \theta_\lambda$ and $\zeta, z \in [0, 1)$, we have:*

$$\mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta(\lambda)}\right] = h_\theta(\zeta), \quad \text{and} \quad \mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta(\lambda)} z^{L_q(\lambda)}\right] = h_\theta(\zeta w^{q,\theta}(z)), \tag{47}$$

with $w^{q,\theta}(z) = g(h_q(z)) + g(0)(z - 1)$.

Proof. We write L_θ for $L_\theta(\lambda)$. Conditioning on the number of children of the lowest branching point and using the branching property of the Galton-Watson trees $\tau_\theta(\lambda)$, we get:

$$\mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta}\right] = g_\theta(0)\zeta + \sum_{k=1}^\infty \mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta}\right]^k \frac{g_\theta^{(k)}(0)}{k!} = g_\theta\left(\mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta}\right]\right) + g_\theta(0)(\zeta-1).$$

This gives the first part of (47). Recall $\mathcal{G}_{q,\theta}(\psi,\lambda)$ defined in Section 6. Using again the branching property, we have:

$$\mathbb{E}\left[z^{L(\mathcal{G}_{q,\theta}(\psi,\lambda))}\right] = g(0)z + g(h_q(z)) - g(0).$$

Then, by Theorem 6.3, we have:

$$\begin{aligned}\mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta}z^{L_q}\right] &= \mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta}z^{\sum_{x\in \mathrm{Lf}(\tau_\theta(\lambda))} L(\mathcal{G}^x_{q,\theta}(\psi,\lambda))}\right]\\ &= \mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta}(g(h_q(z)) + g(0)(z-1))^{L_\theta}\right]\\ &= h_\theta\Big(\zeta(g(h_q(z)) + g(0)(z-1))\Big).\end{aligned}$$

This ends the proof. □

Example 7.6. Assume $\psi(u) = \beta u^2$, with $\beta > 0$, so that $\Theta = \mathbb{R}$ and $q_0 = 0$. Let $\lambda > 0$. We have $\eta = \sqrt{\lambda/\beta}$, $\theta_\lambda = -\eta/2$ and $\Theta^{\psi,\lambda} = [\theta_\lambda, +\infty)$. For $\theta > \theta_\lambda$ and $\zeta \in [0,1)$, we have:

$$\mathbb{E}^{\psi,\lambda}\left[\zeta^{L_\theta(\lambda)}\right] = \frac{\eta+\theta-\sqrt{\theta^2\zeta + (1-\zeta)(\theta+\eta)^2}}{\eta}$$

and for $\theta \neq 0$ (see 55 for $\theta_\lambda < \theta < 0$):

$$\mathbb{E}^{\psi,\lambda}\left[L_\theta(\lambda)\right] = \frac{\eta+2\theta}{2|\theta|}\cdot$$

For $\theta = \theta_\lambda$, we have $g_{\theta_\lambda}(0) = 0$, and the tree $\tau_{\theta_\lambda}(\lambda)$ is a Yule tree and has no leaf (formally, we have $\mathbb{E}^{\psi,\lambda}\left[\zeta^{L_{\theta_\lambda}(\lambda)}\right] = 0$).

8. Ascension time and tree at the ascension time

For convenience, we assume in this Section that ψ is a critical branching mechanism satisfying (H1-3).

8.1. **Ascension process and Ascension time.** Let $\lambda > 0$. Recall θ_λ and $\Theta^{\psi,\lambda}$ defined in Section 6.2. Define the ascension time on $\{M_\lambda \geq 1\}$:

$$A_\lambda = \inf\{\theta \in \Theta^{\psi,\lambda}; \tau_\theta(\lambda) \text{ is a compact tree.}\}, \tag{48}$$

where we make the convention that $\inf\emptyset = \theta_\lambda$. $\mathbb{P}^{\psi,\lambda}$-a.s., we have $A_\lambda \leq 0$. Since, by construction, $\tau_\theta(\lambda)$ is a compact tree if and only if $\mathcal{T}_\theta$ is a compact tree, we have $A_\lambda = \inf\{\theta \in \Theta^{\psi,\lambda} : \sigma_\theta < \infty\}$.

For $\theta \in \Theta$, we set $\bar\theta = \psi^{-1}(\psi(\theta))$, so that $\bar\theta$ is the unique positive number such that:

$$\psi(\bar\theta) = \psi(\theta). \tag{49}$$

By Theorem 6.5 of [2] and its proof, we have for all $\theta \in \Theta^\psi$:

$$\bar\theta - \theta = \psi_\theta^{-1}(0). \tag{50}$$

Recall g_θ defined in (44). And notice $1-\frac{\bar\theta-\theta}{\eta}$ is the minimal solution of the equation $r=g_\theta(r)$. Since $\tau_\theta(\lambda)$ is under $\mathbb{P}^{\psi,\lambda}$ a Galton-Watson tree such the reproduction law has generating function g_θ, we deduce that for $\theta\in(\theta_\lambda,0)$:

$$\mathbb{P}^{\psi,\lambda}(A_\lambda<\theta)=\mathbb{P}^{\psi,\lambda}(\tau_\theta(\lambda) \text{ is compact})=1-\frac{\bar\theta-\theta}{\eta}\cdot \tag{51}$$

Since $d\bar\theta/d\theta=\psi'(\theta)/\psi'(\bar\theta)$, we have for $\theta_\lambda<\theta<0$:

$$\mathbb{P}^{\psi,\lambda}(A_\lambda\in d\theta)=\frac{1}{\eta}\left(1-\frac{\psi'(\theta)}{\psi'(\bar\theta)}\right)d\theta. \tag{52}$$

We give the distribution of the sub-tree at the ascension time. We set $\mathcal{S}_\theta(\lambda)=(\tau_{\theta+q}(\lambda),q\geq 0)$. Thanks to Corollary 4.5 and Theorem 6.1, for $\theta\in\Theta^{\psi,\lambda}$, $\mathcal{S}_\theta(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$ is distributed as $\mathcal{S}_0(\psi_\theta(\eta))$ under $\mathbb{P}^{\psi_\theta,\psi_\theta(\eta)}$.

Proposition 8.1. *Let $\lambda>0$ and $\eta=\psi^{-1}(\lambda)$. For $\theta_\lambda<\theta<0$ and any non-negative measurable function F, we have:*

$$\mathbb{E}^{\psi,\lambda}[F(\mathcal{S}_{A_\lambda}(\lambda))|A_\lambda=\theta]=\frac{\eta\psi'(\bar\theta)}{\psi_\theta(\eta)}\mathbb{E}^{\psi,\lambda}\left[F(\mathcal{S}_\theta(\lambda))\,L_\theta(\lambda)\,\mathbf{1}_{\{L_\theta(\lambda)<\infty\}}\right].$$

Proof. By considering $\mathbb{E}^{\psi,\lambda}\left[F(\mathcal{S}_q)|\tau_q(\lambda)\right]$ instead of $F(\mathcal{S}_q(\lambda))$, one can assume that F is measurable defined on $\mathbb{T}$. Assume $F(T)=0$ if T is non compact. For $\theta_\lambda<q<\theta<0$, we have:

$$\mathbb{E}^{\psi,\lambda}[F(\tau_\theta(\lambda))\mathbf{1}_{\{A_\lambda\geq q\}}]=\mathbb{E}^{\psi,\lambda}[F(\tau_\theta(\lambda))\mathbb{P}^{\psi,\lambda}(\tau_q(\lambda) \text{ is non compact}|\tau_\theta(\lambda))]. \tag{53}$$

We write L_θ for $L_\theta(\lambda)$. On $\{\tau_\theta(\lambda) \text{ compact}\}$ that is $\{L_\theta<\infty\}$, we get that $\tau_q(\lambda)$ is compact if and only if the trees grafted on $\tau_\theta(\lambda)$ to get $\tau_q(\lambda)$, see Theorem 6.3, are compact. Using (51), (42) and notation (44), we get on $\{L_\theta<\infty\}$:

$$\mathbb{P}^{\psi,\lambda}(\tau_q(\lambda) \text{ is non compact}|\tau_\theta(\lambda))=1-g\left(1-\frac{\bar q-q}{\eta}\right)^{L_\theta}. \tag{54}$$

A simple calculation (recall g depends on q) based on the computation of (52) yields on $\{L_\theta<\infty\}$:

$$\begin{aligned}\frac{d}{dq}g\left(1-\frac{\bar q-q}{\eta}\right)^{L_\theta}_{|q=\theta}&=L_\theta\, g\left(1-\frac{\bar q-q}{\eta}\right)^{L_\theta-1}_{|q=\theta}\frac{d}{dq}g\left(1-\frac{\bar q-q}{\eta}\right)_{|q=\theta}\\&=L_\theta\left[\frac{dg}{dq}\left(1-\frac{\bar q-q}{\eta}\right)-g'\left(1-\frac{\bar q-q}{\eta}\right)\frac{1}{\eta}\left(1-\frac{\psi'(q)}{\psi'(\bar q)}\right)\right]_{|q=\theta}\\&=L_\theta\,\frac{\psi'(\bar\theta)}{\psi_\theta(\eta)}\cdot\end{aligned}$$

Then by (53) and (54) and thanks to the regularity of g and $\bar q$ in q, we have:

$$\begin{aligned}\frac{\mathbb{E}^{\psi,\lambda}[F(\tau_\theta(\lambda)),A_\lambda\in d\theta]}{d\theta}&=-\frac{d}{dq}\mathbb{E}^{\psi,\lambda}[F(\tau_\theta(\lambda))\mathbf{1}_{\{A_\lambda\geq q\}}]_{|q=\theta}\\&=\mathbb{E}^{\psi,\lambda}\left[F(\tau_\theta(\lambda))L_\theta\,\frac{\psi'(\bar\theta)}{\psi_\theta(\eta)}\mathbf{1}_{\{L_\theta<\infty\}}\right].\end{aligned}$$

Meanwhile, by Proposition 7.5, we have:

$$\mathbb{E}^{\psi,\lambda}[L_\theta\mathbf{1}_{\{L_\theta<\infty\}}]=\lim_{\zeta\to1-}\frac{\partial}{\partial\zeta}h_\theta(\zeta)=\frac{g_\theta(\eta)(0)}{1-g'_\theta(h_\theta(1-))}=\frac{\psi_\theta(\eta)}{\eta\psi'(\bar\theta)}, \tag{55}$$

where we use the fact that $h_\theta(1-) = 1 - \frac{\bar\theta-\theta}{\eta}$ which is the minimal solution of the equation $r = g_\theta(r)$. Thus, we get:

$$\begin{aligned}\mathbb{E}^{\psi,\lambda}[F(\tau_\theta(\lambda))|A_\lambda \in d\theta] &= \frac{\mathbb{E}^{\psi,\lambda}[F(\tau_\theta(\lambda)), A_\lambda \in d\theta]}{\mathbb{P}^{\psi,\lambda}(A_\lambda \in d\theta)}\\ &= \frac{\mathbb{E}^{\psi,\lambda}[F(\tau_\theta(\lambda))L_\theta \mathbf{1}_{\{L_\theta<+\infty\}}]}{\mathbb{E}^{\psi,\lambda}[L_\theta \mathbf{1}_{\{L_\theta<+\infty\}}]}\\ &= \frac{\eta\psi'(\bar\theta)}{\psi_\theta(\eta)}\,\mathbb{E}^{\psi,\lambda}\left[F(\tau_\theta(\lambda))L_\theta \mathbf{1}_{\{L_\theta<\infty\}}\right].\end{aligned}$$

This ends the proof. □

We give an immediate Corollary.

Corollary 8.2. *Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. For $\theta_\lambda < \theta < 0$ and any non-negative measurable function F, we have, with $\eta_\theta = \eta - \bar\theta + \theta$:*

$$\mathbb{E}^{\psi,\lambda}[F(\mathcal{S}_{A_\lambda}(\lambda))|A_\lambda = \theta] = \frac{\eta_\theta\psi'(\bar\theta)}{\psi_{\bar\theta}(\eta_\theta)}\mathbb{P}^{\psi,\lambda}\left[F(\mathcal{S}_{\bar\theta}(\psi(\eta_\theta)))L_{\bar\theta}(\psi(\eta_\theta))\right].$$

Proof. Then similarly to Proposition 4.6 of [3], we have:

$$\begin{aligned}\mathbb{E}^{\psi,\lambda}[F(\mathcal{S}_{A_\lambda}(\lambda))|A_\lambda = \theta] &= \frac{\eta\psi'(\bar\theta)}{\psi_\theta(\eta)}\mathbb{E}^{\psi,\lambda}\left[F(\mathcal{S}_\theta(\lambda))L_\theta(\lambda)\mathbf{1}_{\{L_\theta(\lambda)<\infty\}}\right]\\ &= \frac{\psi'(\bar\theta)}{\psi_\theta(\eta)}\mathbb{N}^{\psi}\left[F(\mathcal{S}_\theta(\lambda))L_\theta(\lambda)\mathbf{1}_{\{1\le L_\theta(\lambda)<\infty\}}\right]\\ &= \frac{\psi'(\bar\theta)}{\psi_\theta(\eta)}\mathbb{N}^{\psi_\theta}\left[F(\mathcal{S}_0(\psi_\theta(\eta)))L_0(\psi_\theta(\eta))\mathbf{1}_{\{1\le L_0(\psi_\theta(\eta))<\infty\}}\right]\\ &= \frac{\psi'(\bar\theta)}{\psi_\theta(\eta)}\mathbb{N}^{\psi_{\bar\theta}}\left[F(\mathcal{S}_0(\psi_\theta(\eta)))L_0(\psi_\theta(\eta))\mathbf{1}_{\{L_0(\psi_\theta(\eta))\ge 1\}}\right]\\ &= \frac{\psi'(\bar\theta)}{\psi_\theta(\eta)}\mathbb{N}^{\psi}\left[F(\mathcal{S}_{\bar\theta}(\psi(\eta-\bar\theta+\theta)))L_{\bar\theta}(\psi(\eta-\bar\theta+\theta))\mathbf{1}_{\{L_{\bar\theta}(\psi(\eta-\bar\theta+\theta))\ge 1\}}\right]\\ &= \frac{(\eta-\bar\theta+\theta)\psi'(\bar\theta)}{\psi_{\bar\theta}(\eta-\bar\theta+\theta)}\mathbb{P}^{\psi,\lambda}\left[F(\mathcal{S}_{\bar\theta}(\psi(\eta-\bar\theta+\theta)))L_{\bar\theta}(\psi(\eta-\bar\theta+\theta))\right],\end{aligned}$$

where we used Proposition 8.1 for the first equality; definition (25) of $\mathbb{P}^{\psi,\lambda}$ and $\{M_\lambda \ge 1\} = \{L_\theta(\lambda) \ge 1\}$ for the second; Proposition 4.1 for the third; Girsanov transformation (19) and $\psi_\theta(\bar\theta - \theta) = 0$ as well as the fact that the number of the leaves are finite under $\mathbb{N}^{\psi_{\bar\theta}}$ as $\psi_{\bar\theta}$ is sub-critical for the fourth; Proposition 4.1 as well as the equality $\psi_{\bar\theta}(\eta - \bar\theta + \theta) = \psi_\theta(\eta)$ for the fifth and sixth equalities. □

8.2. **An infinite CRT and its pruning.** An infinite CRT was constructed in [2] which, because of (H2) is the Lévy CRT conditioned to have infinite height. Notice that since ψ is critical the event of infinite height is of measure zero. Before recalling its construction, we stress that under $\mathbb{P}^\psi_r$, the root $\emptyset$ belongs to Br_∞ and has mass $\Delta_\emptyset = r$. We identify the half real line $[0, +\infty)$ with a real tree denoted by $[\![0,\infty[\![$ with the null mass measure. We denote by dx the length measure on $[\![0,\infty[\![$. Let $\sum_{i\in I^*} \delta_{(x^*_i, T^{*,i})}$ be a Poisson point measure on $[\![0,\infty[\![\times\mathbb{T}$ with intensity $dx\,\mathbf{N}^{\psi}[d\mathcal{T}]$, with $\mathbf{N}^{\psi}[d\mathcal{T}]$ defined in (12). The infinite CRT from [2] is defined as:

$$\mathcal{T}^* = [\![\emptyset,\infty[\![\circledast_{i\in I^*}(T^{*,i},x_i^*). \tag{56}$$

We denote by $\mathbb{P}^{*,\psi}(d\mathcal{T}^*)$ the distribution of $\mathcal{T}^*$. Following [2] and similarly to the setting in Section 2.9, we consider on $\mathcal{T}^*$ a mark process $M^{\mathcal{T}^*}(d\theta,dy)$ which is a Poisson point measure on $\mathbb{R}_+\times\mathcal{T}^*$ with intensity:

$$\mathbf{1}_{[\emptyset,+\infty)}(\theta)d\theta\left(2\beta\ell^{\mathcal{T}^*}(dy)+\sum_{i\in I^*}\sum_{x\in\mathrm{Br}_\infty(T^{*,i})}\Delta_x\delta_x(dy)\right),$$

with the identification of x_i^* as the root of $\mathcal{T}^{*,i}$. In particular nodes in $[\![\emptyset,\infty[\![$ with infinite degree will be charged by $M^{\mathcal{T}^*}$. For every $x\in\mathcal{T}^*$, we set:

$$\theta^*(x)=\inf\{\theta>0,\ M^{\mathcal{T}^*}([0,\theta]\times[\![\emptyset,x]\!])>0\}.$$

Then we define the pruned tree at time q as $\mathcal{T}_q^*=\{x\in\mathcal{T}^*,\ \theta^*(x)\geq q\}$ with the induced metric, root $\emptyset$ and mass measure the restriction to $\mathcal{T}_q^*$ of the mass measure $\mathbf{m}^{\mathcal{T}^*}$.

Given $\mathcal{T}^*$, let $\mathcal{P}^*(dtdx)=\sum_{j\in J^*}\delta_{(t_j^*,y_j^*)}$ be a Poisson point measure on $[0,\infty)\times\mathcal{T}^*$ with intensity $dt\,\mathbf{m}^{\mathcal{T}^*}(dx)$. For $\theta\geq 0$ and $\lambda>0$, define the pruned sub-tree $\tau_\theta^*(\lambda)$ containing the root and all the ancestors in $\mathcal{T}_\theta^*$ of the marked leaves of $\mathcal{T}^*$:

$$\tau_0^*(\lambda)=\bigcup_{j\in J^*,t_j^*\leq\lambda}[\![\emptyset,y_j^*]\!]\quad\text{and}\quad\tau_\theta^*(\lambda)=\tau_0^*(\lambda)\bigcap\mathcal{T}_\theta^*. \tag{57}$$

We define $\tau_\theta^*(0)=\bigcap_{\lambda>0}\tau_\theta^*(\lambda)$, and notice that $\tau_\theta^*(0)=[\![\emptyset,\infty[\![$ and that it has no leaf. Similarly to (27), we define the mass measure of $\tau_\theta^*(\lambda)$ by:

$$\mathbf{m}^{\tau_\theta^*(\lambda)}=\frac{1}{\psi_\theta(\eta)}\sum_{x\in\mathrm{Lf}(\tau_\theta^*(\lambda))}\delta_x, \tag{58}$$

with $\eta=\psi^{-1}(\lambda)$ and the convention the mass measure is zero if $\lambda=0$.

We have a similar convergence result as Theorem 5.1.

Theorem 8.3. *For all $\theta\geq 0$, we have $\mathbb{P}^{*,\psi}$-a.s.:*

$$\lim_{\lambda\to+\infty}d_{GHP}(\mathcal{T}_\theta^*,\tau_\theta^*(\lambda))=0.$$

Proof. According to [2], there exists a family of random continuous functions $(H^{(a)},a>0)$ with compact support such that: $H^{(a)}$ takes values in $[0,a]$; for all $0<b<a$ and $t\geq 0$, we have:

$$H^{(b)}(t)=H^{(a)}(C_{b,a}^{-1}(t))\quad\text{with}\quad C_{b,a}(s)=\int_0^s\mathbf{1}_{\{H^{(a)}(r)\leq b\}}\,dr;$$

and $\left((\mathcal{T}_\theta^*)^{(a)},a>0\right)$ under $\mathbb{P}^{*,\psi}$ is distributed as $\left(\mathcal{T}^{H^{(a)}},a>0\right)$. Following the proof of Lemma 5.4, we get that for all $a>0$, $\mathbb{P}^{*,\psi}$ a.s.

$$\lim_{\lambda\to+\infty}d^c_{\mathrm{GHP}}\left((\mathcal{T}_\theta^*)^{(a)},(\tau_\theta^*(\lambda))^{(a)}\right)=0.$$

This and the definition of d_{GHP} gives the result. $\square$

Remark 8.4. Similarly to Theorem 3.1, according to the argument in [11], we could reconstruct $\mathcal{T}^*$ from $\tau_0^*(\lambda)$. Recall (30). Conditionally on $\tau_0^*(\lambda)$, $\mathcal{T}_0^*$ is distributed as:

$$\tilde{\tau}_0^*(\lambda) \circledast_{i\in I} (\mathcal{T}_i^*, x_i^*) \circledast_{x\in \mathrm{Br}(\tilde{\tau}_0^*(\lambda))} (\mathcal{T}_x^*, x),$$

with:

- $\tilde{\tau}_0^*(\lambda)$ as $\tau_0^*(\lambda)$ but with 0 as mass measure,
- $\sum_{i\in I} \delta_{(x_i^*, \mathcal{T}_i^*)}$ is a random Poisson point measure on $\tilde{\tau}_0^*(\lambda) \times \mathbb{T}$ with intensity given by $\ell^{\tilde{\tau}_0^*(\lambda)}(dx)\, \mathbf{N}^{\psi_\eta}[d\mathcal{T}]$,
- conditionally on $\sum_{i\in I} \delta_{(x_i^*, \mathcal{T}_i^*)}$, the trees $(\mathcal{T}_x^*, x \in \mathrm{Br}(\tilde{\tau}_0^*(\lambda)))$ are independent with $\mathcal{T}_x^*$ is distributed as:
$$\int \Gamma^{\psi}_{\kappa(x),\lambda}(dr)\, \mathbb{P}_r^{\psi_\eta}[d\mathcal{T}].$$

8.3. **Distribution of the sub-tree of the infinite CRT.** Recall that $\tilde{\tau}_0(\lambda)$ is under $\mathbb{P}^{\psi,\lambda}$ a Galton-Watson tree with distribution $\mathrm{P}^{\psi,\lambda}$. We shall now describe the distribution of $\tilde{\tau}_0^*(\lambda)$ under $\mathbb{P}^{\psi}$, which can be seen as a Galton-Watson tree with distribution $\mathrm{P}^{\psi,\lambda}$ conditionally on the non extinction event.

Let K be an integer-valued random variable with generating function $g_{(\psi,\lambda)}$ defined by (28). Since ψ is critical, we have $g'_{(\psi,\lambda)}(1) = 1$, which implies that $g'_{(\psi,\lambda)}$ itself is the generating function of a integer-valued random variable, say K^*. Since $g'_{(\psi,\lambda)}(0) = 0$, K^* is a.s. positive. Notice that the distribution of K^*+1 is the size-biased distribution of K. Let $(\tau^{k,*}, k \in \mathbb{N}^*)$ be independent random trees distributed as $\tau_0(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$ (that is with distribution $\mathrm{P}^{\psi,\lambda}$ and mass measure given by (27)) independent of K^*. We set:

$$\mathcal{G}^* = \emptyset \circledast_{1\le k\le K^*} (\tau^{k,*}, \emptyset).$$

Theorem 8.5. *Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. Under $\mathbb{P}^{*,\psi}$, $\tau_0^*(\lambda)$ is a rooted real tree distributed as:*

$$[\![\emptyset, \infty[\![\circledast_{i\in I_0^*} (\mathcal{G}^{*,i}, x_i^*),$$

where $\sum_{i\in I_0^*} \delta_{x_i^*}$ is a Poisson point measure on $[\![\emptyset,\infty[\![$ with intensity $\psi'(\eta)dx$ and conditionally on this Poisson point measure, the real trees $(\mathcal{G}^{*,i}, i \in I_0^*)$ are independent and distributed as $\mathcal{G}^*$.

Proof. By construction, thanks to (56), we have:

$$\tau_0^*(\lambda) = [\![\emptyset, \infty[\![\circledast_{i\in I^*} (\tau^{*,i}(\lambda), x_i^*),$$

with $\tau^{*,i}(\lambda) = \bigcup_{j\in J^*, t_j^*\le\lambda, x_i^*\preccurlyeq y_j^*} [\![x_i^*, y_j^*]\!]$ distributed as $\tau_0(\lambda)$ under $\mathbf{N}^{\psi}[d\mathcal{T}]$. The marked Poisson point measure $\sum_{i\in I^*} \mathbf{1}_{\tau^{*,i}(\lambda)\neq\emptyset\}} \delta_{x_i^*}$ is a Poisson point measure on $[\![\emptyset,\infty[\![$ with intensity $\mathbf{N}^{\psi}[M_\lambda \ge 1]\, dx = \psi'(\eta)\, dx$.

Let $I_0^* = \{i \in I^*; \tau^{*,i}(\lambda) \neq \emptyset\}$. The sub-trees $(\tau^{*,i}(\lambda), i \in I_0^*)$ are independent and distributed as $\tau_0(\lambda)$ under $\mathbf{N}^{\psi}[\,\cdot\,|M_\lambda \ge 1]$. Let $N_\emptyset$ be the degree of the root of $\tau_0(\lambda)$. The theorem will be proved once we check that $N_\emptyset$ under $\mathbf{N}^{\psi}[\,\cdot\,|M_\lambda \ge 1]$ is distributed as K^*. Following the proof of Lemma 6.4, we set $h^*(u) = \mathbf{N}^{\psi}\left[u^{N_\emptyset}\mathbf{1}_{\{N_\emptyset\ge 1\}}\right]$, and we have for $u \in [0,1]$:

$$\begin{aligned} h^*(u) &= 2\beta \mathbb{N}^{\psi}[M_\lambda \ge 1]u + \int_{(0,+\infty)} r\pi(dr)\, \mathbb{E}_r^{\psi}\left[u^{N_\emptyset}\mathbf{1}_{\{N_\emptyset\ge 1\}}\right] \\ &= 2\beta\eta u + \int_{(0,+\infty)} r\pi(dr)\, \left(\mathrm{e}^{-r\eta(1-u)} - \mathrm{e}^{-r\eta}\right). \end{aligned}$$

Elementary computations yield $g'_{(\psi,\lambda)}(u) = h^*(u)/h^*(1)$. Thus $N_\emptyset$ under $\mathbf{N}^\psi[\,\cdot\,|M_\lambda \geq 1]$ is distributed as K^*. This ends the proof. $\square$

We give a similar representation formula for $\tau^*_\theta(\lambda)$. Let K^*_θ be an integer-valued random variable with generating function $g'_\theta/g'_\theta(1)$, see definitions (44) and (28). Since $g'_\theta(0) = 0$, K^*_θ is a.s. positive. Notice that the distribution of $K^*_\theta + 1$ is the size-biased distribution of K_θ with generating function g_θ. Let $(\tau^{k,*}_\theta, k \in \mathbb{N}^*)$ be independent random trees distributed as $\tau_\theta(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$ (that is with distribution $\mathrm{P}^{\psi_\theta,\psi_\theta(\eta)}$ and mass measure given by (27)) independent of K^*_θ. We set:

$$\mathcal{G}^*_\theta = \emptyset \circledast_{1\leq k\leq K^*_\theta} (\tau^{k,*}_\theta, \emptyset).$$

Theorem 8.6. *Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. For $\theta > 0$, under $\mathbb{P}^{*,\psi}$, $\tau^*_\theta(\lambda)$ is a rooted real tree distributed as:*

$$[\![\emptyset, E_\theta]\!] \circledast_{i\in I^*_\theta} (\mathcal{G}^{*,i}_\theta, x^*_i),$$

where

- *$[\![\emptyset, E_\theta]\!]$ is a real tree rooted at $\emptyset$ with no branching point and zero mass measure and such that $d(\emptyset, E_\theta)$ is an exponential random variable with parameter $\psi'_\theta(0)$,*
- *$\sum_{i\in I^*_\theta} \delta_{x^*_i}$ is an independent Poisson point measure on $[\![\emptyset, E_\theta]\!]$ with intensity $[\psi'_\theta(\eta) - \psi'_\theta(0)]\, dx$,*
- *conditionally on E_θ and $\sum_{i\in I^*_\theta} \delta_{x^*_i}$, the real trees $(\mathcal{G}^{*,i}, i \in I^*_0)$ are independent and distributed as $\mathcal{G}^*_\theta$.*

Proof. Recall notations of the proof of Theorem 8.5. The distribution of $d(\emptyset, E_\theta)$ is given in [2]. By construction, thanks to (56), we have:

$$\tau^*_0(\lambda) = [\![\emptyset, E_\theta]\!] \circledast_{i\in I^*} (\tau^{*,i}_\theta(\lambda), x^*_i),$$

with $\tau^{*,i}_\theta(\lambda) = \tau^{*,i}(\lambda) \bigcap \mathcal{T}^*_\theta$. Let $N_{\emptyset,\theta}$ (resp. $N'_\emptyset$) be the degree of the root of $\tau_\theta(\lambda)$ (resp. $\tau_0(\psi_\theta(\eta))$). Notice that $\tau^{*,i}_\theta(\lambda)$ is distributed as $\tau_\theta(\lambda)$ under $\mathbf{N}^\psi[d\mathcal{T}, N_\emptyset \geq 1]$ that is as $\tau_0(\psi_\theta(\eta))$ under $\mathbf{N}^{\psi_\theta}[d\mathcal{T}]$. The rate at which sub-trees are grafted on the spine $[\![\emptyset, E_\theta]\!]$ is given by:

$$\mathbf{N}^{\psi_\theta}\left[N'_\emptyset \geq 1\right] = \psi'_\theta(\eta) - \psi'_\theta(0).$$

Then to end the proof, it is enough to check that $N'_\emptyset$ under $\mathbf{N}^{\psi_\theta}[\,\cdot\,|\, N_\emptyset \geq 1]$ is distributed as K^*_θ. Elementary computations give:

$$h^*_\theta(u) = \mathbf{N}^{\psi_\theta}\left[u^{N'_\emptyset}\mathbf{1}_{\{N'_\emptyset\geq 1\}}\right] = \psi_\theta(\eta) - \psi'_\theta(\eta(1-u)),$$

so that $h^*_\theta(u)/h^*_\theta(1) = g'_\theta(u)/g'_\theta(1)$. Thus, $N'_\emptyset$ under $\mathbf{N}^{\psi_\theta}[\,\cdot\,|\, N_\emptyset \geq 1]$ is distributed as K^*_θ. $\square$

We also provide a recursive distribution of the tree $\tau^*_\theta(\lambda)$. Let $a_\theta(\lambda) = \psi'_\theta(0)/\psi'_\theta(\eta) = 1 - g'_\theta(1)$.

Corollary 8.7. *Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. For $\theta > 0$, under $\mathbb{P}^{*,\psi}$, $\tau^*_\theta(\lambda)$ is a rooted real tree distributed as $[\![\emptyset, E_\theta(\lambda)]\!]$ with probability $a_\theta(\lambda)$ and with probability $1 - a_\theta(\lambda)$ as:*

$$[\![\emptyset, E_\theta(\lambda)]\!] \circledast_{0\leq i\leq 1} (\mathcal{G}^{*,i}_\theta, E_\theta(\lambda)),$$

where

- *$[\![\emptyset, E_\theta(\lambda)]\!]$ is a real tree rooted at $\emptyset$ with no branching point and zero mass measure and such that $d(\emptyset, E_\theta(\lambda))$ is an exponential random variable with parameter $\psi'_\theta(\eta)$,*

- *conditionally on $E_\lambda(\theta)$, $\mathcal{G}_\theta^{*,0}$ and $\mathcal{G}^{*,1}$ are independent and distributed respectively as $\mathcal{G}_\theta^*$ and $\tau_\theta^*(\lambda)$.*

Notice that the number of children of $E_\theta(\lambda)$ has generating function $1 - g'_\theta(1) + ug'_\theta(u)$.

Proof. This is a direct consequence of Theorem 8.6, when considering the decomposition of $\tau_\theta^*(\lambda)$ with respect to the lowest branching point and using the branching property. Notice that there is no such branching point (and then $\tau_\theta^*(\lambda)$ is reduce to a spine) if the point measure $\sum_{i\in I_\theta^*} \delta_{x_i^*}$ defined in Theorem 8.6 is zero. This happens with probability $a_\theta(\lambda)$. □

Remark 8.8. Notice that $\tau_\theta^*(\lambda)$ could be obtained from $\tau_0^*(\lambda)$ by a similar pruning procedure as the one defined in Section 6.1.

8.4. **Sub-tree process from the ascension time.** We start with a remark on size-biased discrete Galton-Watson trees.

Remark 8.9. Let $\mathcal{G}$ be a discrete sub-critical Galton-Watson tree starting with one root and with g as generating function of the reproduction law. Let L be the number of leaves of T. We have $\mathbb{E}[L] = g(0)/[1 - g'(1)]$. Let $\mathcal{G}^*$ be distributed as the size-biased distribution of T with respect to L, that is for any non-negative measurable function:

$$\mathbb{E}[F(\mathcal{G}^*)] = \frac{\mathbb{E}[LF(\mathcal{G})]}{\mathbb{E}[L]}.$$

Let $N_\emptyset(t)$ be the number of children of the root of a tree t. For example $N_\emptyset(\mathcal{G})$ has generating function g. The following result can be proved inductively by decomposing the tree with respect to the children of the root.

The distribution of $\mathcal{G}^*$ is characterized as follows. $N_\emptyset(\mathcal{G}^*)$ has generating function $u \to 1 - g'(1) + ug'(u)$. If $N_\emptyset(\mathcal{G}^*) \geq 1$, then label from 1 to $N_\emptyset(\mathcal{G}^*)$ the children of the root and by $\mathcal{G}_i$ the sub-tree attached to children i. Then $(\mathcal{G}_1, \ldots, \mathcal{G}_{N_\emptyset(\mathcal{G}^*)})$ are independent trees; they are distributed as $\mathcal{G}$ but for $\mathcal{G}_I$, for some random index I uniform on $\{1, \ldots, N_\emptyset(\mathcal{G}^*)\}$, which is distributed as $\mathcal{G}^*$.

We denote $\mathcal{S}_\theta^*(\lambda) = (\tau_{\theta+q}^*(\lambda), q \geq 0)$.

Proposition 8.10. *For $\theta > 0$, $\lambda > 0$ and non-negative functionals F, we have:*

$$\frac{\eta\psi'(\theta)}{\psi_\theta(\eta)} \mathbb{E}^{\psi,\lambda}[F(\mathcal{S}_\theta(\lambda))L_\theta(\lambda)] = \mathbb{E}^{*,\psi}[F(\mathcal{S}_\theta^*(\lambda))].$$

Proof. By considering $\mathbb{E}^{\psi,\lambda}[F(\mathcal{S}_\theta)|\tau_\theta(\lambda)]$ instead of $F(\mathcal{S}_\theta(\lambda))$ and $\mathbb{E}^{*,\psi}[F(\mathcal{S}_\theta^*(\lambda))|\tau_\theta^*(\lambda)]$ instead of $F(\mathcal{S}_\theta^*(\lambda))$, one can assume that F is measurable defined on $\mathbb{T}$. Since the life times of all individuals in $\tau_\theta(\lambda)$ and $\tau_\theta^*(\lambda)$ have the same distribution, we only need to consider the distribution of the number of offsprings. This is equivalent to consider the corresponding discrete (or size-biased) Galton-Watson tree. Then the result follows from Remark 8.9. □

Recall that the function $\theta \mapsto \bar{\theta}$ is defined by (49). If $\theta_\lambda \in \Theta^\psi$, then we deduce from (50) that:

$$\bar{\theta}_\lambda - \theta_\lambda = \eta.$$

In particular the function f defined by:

$$f_\lambda(r) = \frac{1}{\eta}\left(1 - \frac{\psi'(r)}{\psi'(\bar{r})}\right)\mathbf{1}_{\{r\in(\theta_\lambda,0)\}}$$

is a probability density. the corresponding cumulative distribution is F_λ defined on $[\theta_\lambda, 0)$ by:

$$F_\lambda(r) = 1 - \frac{\bar{r} - r}{\eta} = \mathbb{P}^{\psi,\lambda}(A_\lambda < r).$$

Proposition 8.11. *Let $\lambda > 0$ and $\eta = \psi^{-1}(\lambda)$. Assume that $\theta_\lambda \in \Theta^\psi$. Under $\mathbb{P}^{*,\psi}$, let U be a random variable with density f_λ and independent of $\mathcal{S}_0^*(\lambda)$. Then $\mathcal{S}_{A_\lambda}(\lambda)$ under $\mathbb{P}^{\psi,\lambda}$ has the same distribution as $\mathcal{S}_{\bar{U}}^*(\psi(\eta F_\lambda(U)))$ under $\mathbb{P}^{*,\psi}$.*

Proof. Using Corollary 8.2, with $\eta_\theta = \eta - \bar{\theta} + \theta$, we get:

$$\begin{aligned}
\mathbb{E}^{\psi,\lambda}[F(\mathcal{S}_{A_\lambda}(\lambda))|A_\lambda = \theta] &= \frac{\eta_\theta \psi'(\bar{\theta})}{\psi_{\bar{\theta}}(\eta_\theta)} \mathbb{E}^{\psi,\lambda}\left[F(\mathcal{S}_{\bar{\theta}}(\psi(\eta_\theta)))L_{\bar{\theta}}(\psi(\eta_\theta))\right] \\
&= \mathbb{E}^{*,\psi}[F(\mathcal{S}_{\bar{\theta}}^*(\psi(\eta_\theta)))].
\end{aligned}$$

Using (52) and $\eta_\theta = \eta F_\lambda(\theta)$, we get:

$$\begin{aligned}
&= \int_{\theta_\lambda}^0 \mathbb{E}^{\psi,\lambda}[F(\mathcal{S}_{A_\lambda}(\lambda))|A_\lambda = \theta]\, \mathbb{P}^{\psi,\lambda}(A_\lambda \in d\theta) \\
&= \int_{\theta_\lambda}^0 \mathbb{E}^{*,\psi}[F(\mathcal{S}_{\bar{\theta}}^*(\psi(\eta_\theta)))] f_\lambda(\theta)\, d\theta \\
&= \int_{\theta_\lambda}^0 \mathbb{E}^{*,\psi}[F(\mathcal{S}_{\bar{\theta}}^*(\psi(\eta F_\lambda(\theta))))] f_\lambda(\theta)\, d\theta \\
&= \mathbb{E}^{*\lambda}[F(\mathcal{S}_{\bar{U}}^*(\psi(\eta F_\lambda(U))))].
\end{aligned}$$

□

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